Operations Research Letters 42 (2014) 332–336

Contents lists available at ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

Using well-solvable minimum cost exact covering for VLSI clock energy minimization

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ARTICLE INFO

Article history:
Received 8 October 2013
Received in revised form
14 February 2014
Accepted 31 May 2014
Available online 10 June 2014

Keywords:
Well-solvable problems
Exact covering
Perfect graph matching
VLSI clocking
Multi-bit flip-flop

ABSTRACT

To save energy of VLSI systems flip-flops (FFs) are grouped in Multi-Bit Flip-Flop (MBFF), sharing a common clock driver. The energy savings strongly depends the grouping. For 2-bit MBFFs the optimal grouping turns into a minimum cost perfect graph matching problem. For k-bit MBFFs the optimal grouping turns into a minimum cost exact k-covering problem. We show that due to their special setting that is based on the FFs' data toggling probabilities, those problems are well-solvable in $O(n \log n)$ time complexity.

1. VLSI energy savings by multi-bit flip-flop grouping

One of the major energy consumers in computing, communication and consumer electronics and other devices is the system's clock signal, typically responsible for 30%–70% of the total switching energy [13]. Flip-flops (FFs) are the heart of digital systems, used to synchronize their operation and store the system's state. To drive the FFs, a clock signal is distributed across the chip through a clocking network. FFs consume most of the clock energy. Within a FF, most of the energy is consumed by its internal clock driver. For simplicity, non-essential VLSI design details are ignored, and the interested reader can find those in any VLSI design textbook (e.g. [15]).

k-bit data is usually stored in k individual FFs, where each of those has its own internal clock drivers. In an attempt to reduce the clock energy, a technique called *Multi-Bit Flip-Flop* (MBFF) has lately been adopted by the VLSI industry [10,4]. A k-bit MBFF combines several FFs integrated in a single entity, such that a common clock driver is used for all the k internal FFs rather than k drivers. The energy savings achieved by using MBFFs is considerable, and may reach up to 20% of the entire system's energy. The savings depend on the average (expected) data toggling probability p of the individual FFs, called *data toggling probability*, *switching probability*, or shortly probability. We use those terms interchangeably. By definition, there is $0 \le p \le 1$, where $p = 0$ when the data is never toggling and $p = 1$ when the data is toggling at every clock cycle. Fig. 1 shows the energy ratio of two and four individual FFs to that of 2-bit and 4-bit MBFFs, respectively. To find the energy savings, we divide the energy difference between k individual FFs and k-bit MBFFs, by the energy of the k individual FFs. For small p it shows savings of $(1.6-1)/1.6 = 35\%$ for $k = 2$ and $(2.2-1)/2.2 = 55\%$ for $k = 4$. For high p the savings is $(1.18-1)/1.18 = 15\%$ for $k = 2$ and $(1.3-1)/1.3 = 23\%$ for $k = 4$. In typical VLSI systems $0 < p < 0.2$, so high savings is expected.

Combining MBFFs with *Data-Driven Clock Gating* (DDCG) considerably increases its energy savings. Ordinarily, FFs receive the clock signal regardless of whether or not their data will toggle in the next cycle. In DDCG the clock signal driving a FF is disabled (gated) when the FF's state is not subject to change (toggle, switch) in the next clock cycle [7,17]. Due to the high hardware overhead involved in generating those signals, it was suggested to group several FFs and derive a joint disabling signal for those. The group size k yielding minimum energy depends on the toggling probabilities [17]. The problem of what FF should belong to what group so that the energy is minimized was studied in [18]. It was shown that under energy model based on the 0/1 toggling

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http://dx.doi.org/10.1016/j.orl.2014.05.010
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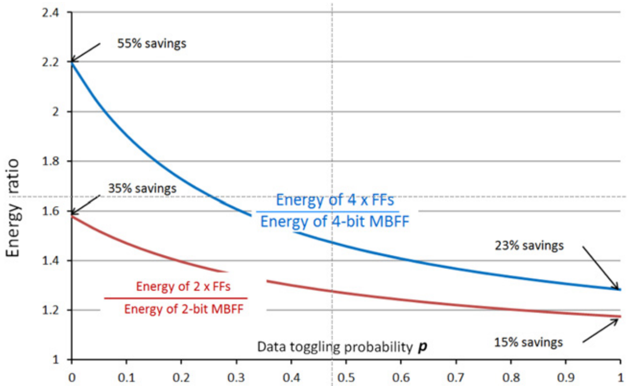

Fig. 1. Energy savings dependency on toggling probabilities of 2-bit and 4-bit MBFFs.

correlation of the FFs, the problem is NP-hard, and a practical heuristic solution based on Minimum Cost Perfect Graph Matching (MCPM) was devised [16].

Applying DDCG in MBFF design methodology was proposed in [5]. However, the grouping in [5] and in other MBFF works [11,19,14] was not aware of the data togging probabilities and correlations, thus a big amount of potential energy savings was left untreated. The work in [16] used toggling correlation to derive the optimal FFs grouping for DDCG. It required huge data of 0/1 toggling vectors of all the FFs, obtained by simulations, which is a serious design burden. Furthermore, the corresponding optimization problem is NP-hard as mentioned before, and heuristic solution was thus proposed.

In this paper we simplify the optimal grouping formulation by considering FF probabilities rather than their 0/1 toggling vectors. The simplification implies an optimization that is a kind of minimal cost exact k-covering problem, where for $k = 2$ it turns into MCPM, formulated as follows. Given n real numbers (data toggling probabilities of FFs) $p_i \in [0,1]$, n even, $1 \le i \le n$, $p_1 \le p_2 \le \cdots \le p_n$, find a perfect matching (s_j, t_j), $1 \le j \le n/2$, of the integers $1, 2, \ldots, n$, minimizing the following energy loss expression (discussed in Section 3)

$$\sum_{j=1}^{n/2} p_{s_j}\left(1-p_{t_j}\right)+p_{t_j}\left(1-p_{s_j}\right).$$

For $k > 2$, let n be an integer multiple of k. Consider the partitioning (called also exact k-covering) of $\{1,2,\ldots,n\}$ into n/k disjoint subsets $S^j \subset \{1,2,\ldots,n\}$, $\left|S^j\right| = k$, $\bigcup_j S^j = \{1,2,\ldots,n\}$. Denote by $\left(A_m^j, B_m^j\right)$ the partitioning of S^j $1 \le m \le \binom{k}{r}$ such that $\left|A_m^j\right| = k-r$, $\left|B_m^j\right| = r$, $S^j = A_m^j \cup B_m^j$ and $A_m^j \cap B_m^j = \varnothing$. The minimum cost exact k-covering problem (discussed in Section 4) is to find those S^j, $1 \le j \le n/k$, minimizing the energy loss expression

$$\sum_{j=1}^{n/k}\sum_{r=1}^{k-1} r \sum_{m=1}^{\binom{k}{r}} \prod_{s\in A_m^j} p_s \prod_{t\in B_m^j}\left(1-p_t\right).$$

While the minimum cost exact covering problem is NP-hard in general [9], we prove that in our special setting it is well-solvable, requires only sorting. Well-solvable cases of hard combinatorial optimization problems are well-known and have been studied by many works. Just a few to mention are special cases of Traveling Salesman Problems (TSP) [2], Steiner tree problems [1], Quadratic Assignment Problems (QAP) [3] and Bounded Knapsack Problem (BKP) [6]. Well-solvable quadratic assignment problems was used for VLSI interconnect design optimization [8].

The rest of the paper is organized as follows. Section 2 presents the MBFF energy consumption model. Section 3 shows that the 2-bit MBFF optimal pairing is MCPM problem, well-solvable by sorting of FFs' toggling probabilities. Section 4 shows that the k-bit MBFF optimal grouping is a minimum cost exact k-covering problem, that it is also well-solvable by sorting of FFs' toggling probabilities.

2. Energy consumption of multi-bit flip-flops

The energy E_1 consumed by an ordinary 1-bit FF grows with its toggling probability p as follows:

$$E_1(p) = \alpha_1 + \beta_1 p. \tag{1}$$

The parameter α_1 is the energy of the FF's internal clock driver, and the parameter β_1 is the energy of data toggling. For 2-bit MBFF there are three possible scenarios: none of the FFs toggle, a single FF toggles and both FFs toggle. Assuming toggling independency, the energy consumption E_2 is

$$\begin{aligned} E_2(p) &= \alpha_2(1-p)^2 + 2(\alpha_2+\beta_2)p(1-p) + (\alpha_2+2\beta_2)p^2 \\ &\equiv \alpha_2 + 2\beta_2 p. \end{aligned} \tag{2}$$

The parameter α_2 is the energy of the internal clock driver which drives the two FFs, and the parameter β_2 is of data toggling energy of one bit in the 2-bit MBFF. The energy savings factor $2E_1(p)/E_2(p)$ is shown in Fig. 1. Obviously, the lower the data toggling probability is, the higher the savings factor is.

For the general case of k-bit MBFF, let α_k be the energy of the MBFF's internal clock driver driving its k FFs, and let the parameter β_k be the data toggling energy of one bit in the k-bit MBFF. Considering all the combinations of toggling FFs, the energy consumption E_k is

$$E_k(p) = \sum_{j=0}^{k}(\alpha_k + j\beta_k)\binom{k}{j}p^j(1-p)^{k-j}. \tag{3}$$

Rearrangement of (3) yields

$$\begin{aligned} E_k(p) &= \alpha_k \sum_{j=0}^{k}\binom{k}{j}p^j(1-p)^{k-j} + \beta_k\sum_{j=0}^{k}\binom{k}{j}jp^j(1-p)^{k-j} \\ &= \alpha_k + k\beta_k p. \end{aligned} \tag{4}$$

The equality $\sum_{j=0}^{k}\binom{k}{j}jp^j(1-p)^{k-j} = kp$ in (4) follows from $j\binom{k}{j} = k\binom{k-1}{j-1}$. The energy savings factor $4E_1(p)/E_4(p)$ is shown in Fig. 1.

3. Optimal FF grouping of 2-bit MBFF

Let FF_i and FF_j toggle independently of each other with probabilities p_i and p_j, respectively. We denote by $FF_{(i,j)}$ their grouping (pairing) in the formation of a 2-bit MBFF. Similar to (2), the energy $E_{(i,j)}$ consumed by $FF_{(i,j)}$ is $E_{(i,j)} = \alpha_2 + \beta_2(p_i + p_j)$. For FF_i, FF_j, FF_k and FF_l, paired in two MBFFs $FF_{(i,j)}$ and $FF_{(k,l)}$, the energy consumption is $E_{(i,j)} + E_{(k,j)} = 2\alpha_2 + \beta_2(p_i + p_j + p_k + p_l)$, which is independent of the pairing.

Pairing considerably affects the energy consumption when DDCG is applied. Recall that with DDCG the clock pulse is disabled when the data of a FF will not change (toggle) in the next clock cycle. Since in MBFF the clock signal is common to all FFs, when none of FF_i and FF_j is toggling, the clock pulse of $FF_{(i,j)}$ is disabled and its clock driver does not waste any energy. When both FF_i and FF_j are toggling, the clock pulse of $FF_{(i,j)}$ is enabled and the energy of the clock driver is fully useful, hence no waste occurs. Energy waste occurs when one FF is toggling, while its counterpart does not. There, the common clock pulse is enabled and is driving both FFs, whereas only one needs it, thus causing a waste $W_{(i,j)}$ of half of the clock driver energy,

$$W_{(i,j)} = \frac{\alpha_2}{2}\left[p_j\left(1-p_i\right)+p_i\left(1-p_j\right)\right] = \frac{\alpha_2}{2}\left(p_i+p_j-2p_ip_j\right). \quad (5)$$

We are interested in the minimization of $W_{(i,j)}$. Applying DDCG to $\mathrm{FF}_{(i,j)}$ and $\mathrm{FF}_{(k,l)}$, the following energy waste results in

$$\begin{aligned} W_{(i,j)} + W_{(k,l)} &= \frac{\alpha_2}{2}\left[p_i\left(1-p_j\right)+p_j\left(1-p_i\right)\right. \\ &\quad \left. + \; p_k\left(1-p_l\right)+p_l\left(1-p_k\right)\right] \\ &= \frac{\alpha_2}{2}\left[p_i+p_j+p_k+p_l-2\left(p_ip_j+p_kp_l\right)\right]. \quad (6) \end{aligned}$$

While the linear term of the right-hand side (6) is independent of the pairing, the quadratic term does. $W_{(i,j)} + W_{(k,l)}$ is minimized when $p_ip_j + p_kp_l$ is maximized.

Lemma 1. *Given $p_i \le p_j \le p_k \le p_l$, the pairing $\left\{FF_{(i,j)}, FF_{(k,l)}\right\}$ is optimal.*

Proof. It follows that $\left(W_{(i,j)} + W_{(k,l)}\right) - \left(W_{(i,k)} + W_{(j,l)}\right) = -\alpha_2 \left(p_i - p_l\right)\left(p_j - p_k\right)/2 \le 0$. Similarly, $\left(W_{(i,j)} + W_{(k,l)}\right) - \left(W_{(i,l)} + W_{(j,k)}\right) = -\alpha_2\left(p_i - p_k\right)\left(p_j - p_l\right)/2 \le 0$. ■

Let n be even (odd n is discussed later) and $\mathbf{P} : \left\{\mathrm{FF}_{(s_i,t_i)}\right\}_{i=1}^{n/2}$ be a pairing of $\mathrm{FF}_1, \mathrm{FF}_2, \ldots, \mathrm{FF}_n$ in $n/2$ 2-bit DDCG MBFFs. The following energy waste $\mathbf{W}(\mathbf{P})$ results in

$$\mathbf{W}(\mathbf{P}) = \sum_{i=1}^{n/2} W_{(s_i,t_i)} = \frac{\alpha_2}{2}\sum_{i=1}^{n/2}\left(1-p_{s_i}\right)\left(1-p_{t_i}\right). \quad (7)$$

It follows from (6) that $\mathbf{W}(\mathbf{P})$ is minimized when $\sum_{i=1}^{n/2} p_{s_i}p_{t_i}$ is maximized. The optimal pairing could be found in polynomial time by applying a MCPM [12] to the n-vertex complete weighted graph, the vertices of which are FF_i and its edge weights are p_ip_j, $1 \le i < j \le n$. The observation made for pairing of four FFs hints that optimal pairing should prefer FFs with close switching probabilities to reside in the same MBFF. The generalization for pairing of n FFs is subsequently discussed, proving that it can be found in $O(n \log n)$ time complexity by sorting.

Theorem 1. *Let n be even and let $FF_1, FF_2, \ldots, FF_n$ be ordered such that $p_1 < p_2 < \cdots < p_n$. The pairing $\mathbf{P} : \left\{FF_{(2i-1,2i)}\right\}_{i=1}^{n/2}$ minimizes $\mathbf{W}(\mathbf{P})$ given in (7).*

Proof. Eq. (7) shows that $W_{(i_1,i_2)}$ is independent of the order of the FFs within a pair. It is therefore assumed w.l.o.g that for a pair (i_1, i_2) there is $p_{i_1} < p_{i_2}$. By (7) $\mathbf{W}(\mathbf{P})$ is independent of the order of the pairs in the summation, so $\mathbf{P}$ is assumed w.l.o.g to be increasingly ordered such that (i_1, i_2) precedes (j_1, j_2) iff $p_{i_1} < p_{j_1}$, $1 \le i_1, j_1 \le n/2$ and $i_1 \ne j_1$. Assume in contrary that there is an increasingly ordered pairing $\mathbf{P}' : \left\{\mathrm{FF}_{(s_{2i-1},s_{2i})}\right\}_{i=1}^{n/2}$, $\mathbf{P}' \ne \mathbf{P}$, minimizing $\mathbf{W}$. Let us compare the $n/2$ pairs of $\mathbf{P}$ with those of $\mathbf{P}'$ by their order, namely $\mathrm{FF}_{(2i-1,2i)} \in \mathbf{P}$ with $\mathrm{FF}_{(s_{2i-1},s_{2i})} \in \mathbf{P}'$, $1 \le i \le n/2$. Let $\mathrm{FF}_{(2j-1,2j)} \in \mathbf{P}$ and $\mathrm{FF}_{(s_{2j-1},s_{2j})} \in \mathbf{P}'$ be the first unmatched pairs. It follows from the increasing order of the pairs that $s_{2j-1} = 2j-1$ and $s_{2j} > 2j$.

Consider the pair $\mathrm{FF}_{(2j,t)} \in \mathbf{P}'$ which follows $\mathrm{FF}_{(2j-1,s_{2j})} \in \mathbf{P}'$. Let us derive a pairing $\mathbf{P}''$ from $\mathbf{P}'$ by exchanging $\mathrm{FF}_{s_{2j}}$ with FF_{2j}. Assume w.l.o.g that $s_{2j} < t$. The pairing $\mathbf{P}'$ and $\mathbf{P}''$ thus differ on $\left\{\mathrm{FF}_{(2j-1,s_{2j})}, \mathrm{FF}_{(2j,t)}\right\} \subset \mathbf{P}'$ and $\left\{\mathrm{FF}_{(2j-1,2j)}, \mathrm{FF}_{(s_{2j},t)}\right\} \subset \mathbf{P}''$, and are identical on the rest $n/2-2$ pairs. The inequality $\mathbf{w}\left(\mathbf{P}''\right) - \mathbf{w}\left(\mathbf{P}'\right) < 0$ follows from Lemma 1, thus concluding that $\mathbf{P}$ is optimal. ■

The time complexity of finding the optimal MBFF pairing is $O(n \log n)$ since only sorting of FFs' toggling probabilities is required. In case of odd n we could artificially add a never toggling FF, hence $p = 0$. Theorem 1 will apply, and the optimal pairing yields $\mathrm{FF}_{(2,3)}, \ldots, \mathrm{FF}_{(n-1,n)}$, whereas FF_1 will stay unpaired.

4. Optimal FF grouping of *k*-bit MBFF

The hardware overhead involved in DDCG may sometimes make its application questionable for groups comprising two FFs. It has been shown in [17] that DDCG is very useful for groups of three and more FFs, depending on their toggling probabilities. We subsequently analyze the case of k-bit MBFFs. Let $\mathrm{FF}_{(i_1,\ldots,i_k)}$ denote a k-bit MBFF comprising $\mathrm{FF}_{i_1}, \ldots, \mathrm{FF}_{i_k}$ and consider its energy waste $W_{(i_1,\ldots,i_k)}$. When none of its underlying FFs is toggling, its DDCG disables the clock pulse, so energy is not wasted. Other than that DDCG enables the clock pulse. When all the FFs are toggling, the clock pulse is anyway required by all the FFs, so there is no energy waste. A waste occurs when $k-r$, $1 \le r \le k-1$, of the FFs are toggling, while r are not. There are $\binom{k}{r}$ events of this kind and they are pairwise distinct. Since the clock pulse drives r non-toggling FFs, the energy waste is $\alpha_k r/k$ multiplied by the probability of that event. For each $1 \le m \le \binom{k}{r}$ we split $\mathrm{FF}_{i_1}, \ldots, \mathrm{FF}_{i_k}$ into A_m and B_m, the indices of the toggling and non-toggling FFs, respectively, $A_m \cup B_m = \{i_1, \ldots, i_k\}$, $A_m \cap B_m = \varnothing$, $|A_m| = k-r$ and $|B_m| = r$. The corresponding energy waste is therefore

$$W_{(i_1,\ldots,i_k)} = \frac{\alpha_k}{k}\sum_{r=1}^{k-1} r \sum_{m=1}^{\binom{k}{r}} \prod_{s\in A_m} p_s \prod_{t\in B_m}\left(1-p_t\right). \quad (8)$$

Consider $n = 2k$ FFs $\mathrm{FF}_{i_1}, \ldots, \mathrm{FF}_{i_{2k}}$, ordered such that $p_{i_1} \le p_{i_2} \le \cdots \le p_{i_{2k-1}} \le p_{i_{2k}}$. We subsequently generalize Lemma 1 for $k > 2$. It is shown that the minimum energy waste occurs for the grouping $\{(i_1, \ldots, i_k), (i_{k+1}, \ldots, i_{2k})\}$, namely, the k FFs with the smaller probabilities in one group while the k FFs with the larger probabilities in the other.

Lemma 2. *Given $2k$ FFs $FF_{i_1}, \ldots, FF_{i_{2k}}$ satisfying $p_{i_1} \le p_{i_2} \le \cdots \le p_{i_{2k-1}} \le p_{i_{2k}}$. The grouping $\{(i_1, \ldots, i_k), (i_{k+1}, \ldots, i_{2k})\}$ minimizes the energy waste.*

Proof. Assume that there exists a grouping $\{(j_1, \ldots, j_k), (j_{k+1}, \ldots, j_{2k})\}$ such that $W_{(j_1,\ldots,j_k)} + W_{(j_{k+1},\ldots,j_{2k})}$ is minimal, but $\{j_1, \ldots, j_k\} \ne \{i_1, \ldots, i_k\}$ (and hence $\{j_{k+1}, \ldots, j_{2k}\} \ne \{i_{k+1}, \ldots, i_{2k}\}$). We split the indices of the smaller and larger probabilities, $L = \{i_1, \ldots, i_k\}$ and $H = \{i_{k+1}, \ldots, i_{2k}\}$, respectively, such that $L = L' \cup L''$, $L' \cap L'' = \varnothing$, $H = H' \cup H''$ and $H' \cap H'' = \varnothing$, as follows:

$$\begin{aligned} L' &= \{i_1, \ldots, i_k\} \cap \{j_1, \ldots, j_k\} \\ L'' &= \{i_1, \ldots, i_k\} \cap \{j_{k+1}, \ldots, j_{2k}\} \\ H' &= \{i_{k+1}, \ldots, i_{2k}\} \cap \{j_1, \ldots, j_k\} \\ H'' &= \{i_{k+1}, \ldots, i_{2k}\} \cap \{j_{k+1}, \ldots, j_{2k}\}. \end{aligned} \quad (9)$$

For every m, $1 \le m \le \binom{k}{r}$, we further split L', L'', H', and H'' according to their corresponding $k-r$ toggling FFs A_m^*, and r non-toggling FFs B_m^*, as follows:

$$\begin{aligned} L' &= A_m^{L'} \cup B_m^{L'}, \qquad A_m^{L'} \cap B_m^{L'} = \varnothing; \\ L'' &= A_m^{L''} \cup B_m^{L''}, \qquad A_m^{L''} \cap B_m^{L''} = \varnothing; \\ H' &= A_m^{H'} \cup B_m^{H'}, \qquad A_m^{H'} \cap B_m^{H'} = \varnothing; \\ H'' &= A_m^{H''} \cup B_m^{H''}, \qquad A_m^{H''} \cap B_m^{H''} = \varnothing. \end{aligned} \quad (10)$$

Consider the energy wastes occurring by the grouping $\{(j_1,\ldots,j_k),(j_{k+1},\ldots,j_{2k})\}$ and the grouping $\{(i_1,\ldots,i_k),(i_{k+1},\ldots,i_{2k})\}$. Substitution of (10) in (8) yields the following expressions

$$W_{(j_1,\ldots,j_k)}+W_{(j_{k+1},\ldots,j_{2k})}$$
$$=\frac{\alpha_k}{k}\sum_{r=1}^{k-1}r\sum_{m=1}^{\binom{k}{r}}\Bigg[\overbrace{\prod_{s\in A_m^{L'}}p_s}^{(a_m)}\overbrace{\prod_{s\in A_m^{H'}}p_s}^{(b_m)}\overbrace{\prod_{t\in B_m^{L'}}(1-p_t)}^{(c_m)}\overbrace{\prod_{t\in B_m^{H'}}(1-p_t)}^{(d_m)}$$
$$+\overbrace{\prod_{s\in A_m^{L''}}p_s}^{(e_m)}\overbrace{\prod_{s\in A_m^{H''}}p_s}^{(f_m)}\overbrace{\prod_{t\in B_m^{L''}}(1-p_t)}^{(g_m)}\overbrace{\prod_{t\in B_m^{H''}}(1-p_t)}^{(h_m)}\Bigg].\tag{11}$$

$$W_{(i_1,\ldots,i_k)}+W_{(i_{k+1},\ldots,i_{2k})}$$
$$=\frac{\alpha_k}{k}\sum_{r=1}^{k-1}r\sum_{m=1}^{\binom{k}{r}}\Bigg[\overbrace{\prod_{s\in A_m^{L'}}p_s}^{(a_m)}\overbrace{\prod_{s\in A_m^{L''}}p_s}^{(e_m)}\overbrace{\prod_{t\in B_m^{L'}}(1-p_t)}^{(c_m)}\overbrace{\prod_{t\in B_m^{L''}}(1-p_t)}^{(g_m)}$$
$$+\overbrace{\prod_{s\in A_m^{H'}}p_s}^{(b_m)}\overbrace{\prod_{s\in A_m^{H''}}p_s}^{(f_m)}\overbrace{\prod_{t\in B_m^{H'}}(1-p_t)}^{(d_m)}\overbrace{\prod_{t\in B_m^{H''}}(1-p_t)}^{(h_m)}\Bigg].\tag{12}$$

The partial products in (11) and (12) are identified by appropriate symbols. Using those symbols, we obtain

$$\left(W_{(i_1,\ldots,i_k)}+W_{(i_{k+1},\ldots,i_{2k})}\right)-\left(W_{(j_1,\ldots,j_k)}+W_{(j_{k+1},\ldots,j_{2k})}\right)$$
$$=\frac{\alpha_k}{k}\sum_{r=1}^{k-1}r\sum_{m=1}^{\binom{k}{r}}[(a_me_mc_mg_m+b_mf_md_mh_m)$$
$$-(a_mb_mc_md_m+e_mf_mg_mh_m)]$$
$$=\frac{\alpha_k}{k}\sum_{r=1}^{k-1}r\sum_{m=1}^{\binom{k}{r}}(a_mc_m-f_mh_m)(e_mg_m-b_md_m)\le 0.\tag{13}$$

The inequality in (13) follows since the terms comprising the product a_mc_m are all smaller than those comprising f_mh_m, while the terms comprising the product e_mg_m are all larger than those comprising b_md_m. We conclude that the grouping $\{(i_1,\ldots,i_k),(i_{k+1},\ldots,i_{2k})\}$ minimizes the energy waste. ■

Assume that n is divisible by k (the case where n is not an integer multiple of k is discussed later). The k-bit grouping of n FFs extends the 2-bit case discussed in Section 3. We subsequently show that the optimal k-bit MBFF grouping can be found in $O(n\log n)$ time complexity by sorting. Let $\mathbf{P}$: $\left\{\left(s_{k(i-1)+1},\ldots,s_{ki}\right)\right\}_{i=1}^{n/k}$ be a grouping of $FF_1, FF_2,\ldots,FF_n$ in n/k k-bit DDCG MBFFs, and let $W_{\left(s_{k(i-1)+1},\ldots,s_{ki}\right)}$ be the energy waste of $FF_{\left(s_{k(i-1)+1},\ldots,s_{ki}\right)}$ given in (8). The total energy waste $\mathbf{W}(\mathbf{P})$ is

$$\mathbf{W}(\mathbf{P})=\sum_{i=1}^{n/k}W_{\left(s_{k(i-1)+1},\ldots,s_{ki}\right)}.\tag{14}$$

The optimal grouping implies a minimal cost exact k-covering problem that is NP-hard in general [9]. In the setting of our problem though, where the costs of the groups are obtained by sum of probabilities products, it is well-solvable, as subsequently shown.

Theorem 2. *Let n be divisible by k and let $FF_1, FF_2,\ldots,FF_n$ be ordered such that $p_1<p_2<\cdots<p_n$. The grouping* $\mathbf{P}$: $\{(k(i-1)+1,\ldots,ki)\}_{i=1}^{n/k}$ *minimizes* $\mathbf{W}(\mathbf{P})$ *given in (14).*

Proof. Eq. (8) shows that $W_{\left(s_{k(i-1)+1},\ldots,s_{ki}\right)}$ is independent of the order of the FFs within a k-bit group. It is therefore assumed w.l.o.g that there is $s_{k(i-1)+1}<s_{k(i-1)+2}<\cdots<s_{ki}$. Since $\mathbf{W}$ in (14) is independent of the order of the terms in the summation, $\mathbf{P}$ is assumed w.l.o.g to be increasingly ordered such that $\left(s_{k(i-1)+1},\ldots,s_{ki}\right)$ precedes $\left(s_{k(j-1)+1},\ldots,s_{kj}\right)$ iff $s_{k(i-1)+1}<s_{k(j-1)+1}$, for $1\le i,j\le n/k$ and $i\ne j$.

Assume in contrary that there is an increasingly ordered grouping $\mathbf{P}'$: $\left\{\left(s_{k(i-1)+1},\ldots,s_{ki}\right)\right\}_{i=1}^{n/k}$, $\mathbf{P}'\ne\mathbf{P}$, minimizing $\mathbf{W}$. Let us compare the n/k groups of $\mathbf{P}$ with those of $\mathbf{P}'$ by their order, namely $(k(i-1)+1,\ldots,ki)\in\mathbf{P}$ with $\left(s_{k(i-1)+1},\ldots,s_{ki}\right)\in\mathbf{P}'$, $1\le i\le n/k$. Let $(k(j-1)+1,\ldots,kj)\in\mathbf{P}$ and $\left(s_{k(j-1)+1},\ldots,s_{kj}\right)\in\mathbf{P}'$ be the first unmatched groups. It follows from the increasing order of $\mathbf{P}$ and $\mathbf{P}'$ that $s_{k(j-1)+1}=k(j-1)+1$, while for the rest indices of the first unmatched groups there is $\left\{s_{k(j-1)+2},\ldots,s_{kj}\right\}\ne\{k(j-1)+2,\ldots,kj\}$. Consider the group $\left(s_{kj+1},\ldots,s_{k(j+1)}\right)\in\mathbf{P}'$ succeeding $\left(s_{k(j-1)+1},\ldots,s_{kj}\right)\in\mathbf{P}'$. By the increasing order of the indices in a group, $s_{kj+1}\in\{k(j-1)+2,\ldots,kj\}$, as otherwise $(k(j-1)+1,\ldots,kj)\in\mathbf{P}$ and $\left(s_{k(j-1)+1},\ldots,s_{kj}\right)\in\mathbf{P}'$ would have been be identical.

Let us derive a grouping $\mathbf{P}''$ from $\mathbf{P}'$ by rearranging the $2k$ indices of $\left\{s_{k(j-1)+1},\ldots,s_{kj},s_{kj+1},\ldots,s_{k(j+1)}\right\}$ such that the k smaller ones reside in a group $\left(t_{k(j-1)+1},\ldots,t_{kj}\right)\in\mathbf{P}''$, while the k larger indices reside in a group $\left(t_{kj+1},\ldots,t_{k(j+1)}\right)\in\mathbf{P}''$, where the indices are renamed appropriately. Other than $\left\{\left(s_{k(j-1)+1},\ldots,s_{kj}\right),\left(s_{kj+1},\ldots,s_{k(j+1)}\right)\right\}\subset\mathbf{P}'$ and $\{(t_{k(j-1)+1},\ldots,t_{kj}),\left(t_{kj+1},\ldots,t_{k(j+1)}\right)\}\subset\mathbf{P}''$, $\mathbf{P}'$ and $\mathbf{P}''$ are identical on the rest $n/k-2$ groups. It then follows by Lemma 2 that $\mathbf{w}\left(\mathbf{P}''\right)-\mathbf{w}\left(\mathbf{P}'\right)<0$, which contradicts that the energy wasted by $\mathbf{P}'$ is minimal, thus concluding that $\mathbf{P}$ is optimal. ■

The time complexity of finding the optimal k-bit MBFF grouping is $O(n\log n)$ since only sorting of FFs' toggling probabilities is required. If n is not an integer multiple of k, we could artificially add $r=n-n \bmod k$ never toggling FFs, and their toggling probability is therefore $p=0$. The problem comprising $n+r$ FFs thus obtained obeys Theorem 2. The artificial FFs $FF_1, FF_2,\ldots,FF_{k-r}$ will then be grouped in a $(k-r)$-bit MBFF, whereas the rest $n-k+r$ FFs will be grouped in $\lfloor n/k\rfloor$ k-bit MBFFs. It should be noted that Theorem 2 is still valid for $p_1\le p_2\le\cdots\le p_n$, where any equality can be arbitrarily resolved.

Acknowledgments

The authors are grateful for the corrections and useful comments and suggestions made by the reviewers.

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