C

Boolean Switching algebra

We have seen that arithmetic and other data can be represented by 2-value (binary). We'll use the symbols 0 and 1.

- Boolean algebra - George Boole, 19 century
- Analysis of digital systems by using Boolean algebra - Claude Shannon 1938
- Every Boolean function ($\{0,1\} \to \{0,1\}$) can be expressed by three operators AND, OR, NOT.

Axioms of Boolean algebra

Boolean algebra - set of elements B with two binary operators $\{+\}$ and $\{\circ\}$ and a unary operator $\{'\}$.

① B contains at least two elements a, b $a \neq b$.

② Closure: $\forall a, b \in B$

$$a+b \in B, \quad a \cdot b \in B$$

③ Commutative laws: $\forall a, b \in B$

$$a+b = b+a, \quad a \cdot b = b \cdot a$$

④ Identities existence

For $\{+\}$ there exists an element called 0 (zero) such that

$$a+0=0 \quad \forall a \in B$$

For $\{\cdot\}$ there exists an element called 1 (one) such that

$$a \cdot 1 = a \quad \forall a \in B$$

⑤ Distribution laws: $\forall a, b, c \in B$

$$a+(b \cdot c) = (a+b) \cdot (a+c)$$

$a \cdot (b+c) = a \cdot b + a \cdot c$

⑥ Complement existence

$\forall\, a \in B\ \exists\, a' \in B$, called complement, (unique)

such that

$$a + a' = 1$$

$$a \cdot a' = 0$$

Principle of duality

all axioms come in pairs, where one can be obtained from the other by

$$0 \Longleftrightarrow 1 \;,\; \{+\} \Longleftrightarrow \{\circ\}$$

⑤ Distribution law

$$a + (b \circ c) = (a + b) \circ (a + c)$$

$$\updownarrow \quad \updownarrow \qquad\qquad \updownarrow \quad \updownarrow \quad \updownarrow$$

$$a \circ (b + c) = (a \circ b) + (a \circ c)$$

⑥ Complement

$$a + a' = 1$$

$$\updownarrow \qquad \updownarrow$$

$$a \circ a' = 0$$

Basic Theorems of Boolean Algebra

Theorem: $\forall a \in B \quad a+a=a \, , \; aa=a$

Proof: ① $a+a \underset{\substack{\uparrow \\ \text{axiom identity}}}{=} (a+a)\cdot 1 \underset{\substack{\uparrow \\ \text{axiom complement}}}{=} (a+a)(a+a') =$

$$\underset{\substack{\uparrow \\ \text{axiom distribution}}}{=} a + a'a \underset{\substack{\uparrow \\ \text{axiom complement}}}{=} a + 0 \underset{\substack{\nwarrow \\ \text{axiom identity}}}{=} a$$

② $a \cdot a = a$ follows from duality

Theorem: $\forall a \in B \quad a+1 = 1 \, , \; a \cdot 0 = 0$

Proof: ① $a+1 \underset{\substack{\uparrow \\ \text{identity}}}{=} (a+1)\cdot 1 \underset{\substack{\uparrow \\ \text{complement}}}{=} (a+1)\cdot(a+a') =$

$$\underset{\substack{\uparrow \\ \text{distribution}}}{=} a + 1a' \underset{\substack{\uparrow \\ \text{identity + comutataire}}}{=} a + a' \underset{\substack{\uparrow \\ \text{complement}}}{=} 1$$

② $a \cdot 0 = 0$ follows from duality

Theorem (involution): $(a')' = a$

Proof

$$(a')(a')' \underset{\substack{\uparrow \\ \text{comp axiom}}}{=} 0 \qquad (a')a \underset{\substack{\uparrow \\ \text{comp axiom}}}{=} 0 \Rightarrow \underset{\substack{\nwarrow \\ \text{comp uniqueness}}}{(a')' = a}$$

Theorem (absorption) $\forall a, b \in B$, $a + ab = a$, $a \cdot (a+b) = a$

Proof: ① $a + ab \underset{\text{identity}}{=} a \cdot 1 + a \cdot b \underset{\text{distribution}}{=} a(1+b) =$

$\underset{\text{theorem}}{=} a \cdot 1 \underset{\text{identity}}{=} a$

② $a \cdot (a+b) = a$ follows from duality

Theorem (associative) $\forall a, b, c \in B$

$$a + (b+c) = (a+b) + c, \quad a(bc) = (ab)c$$

Theorem (concensus): $\forall x, y, z \in B$

$$xy + x'z + yz = xy + x'z$$

Proof: $xy + x'z + yz =$

$= xy(z+z') + x'(y+y')z + (x+x')yz =$

$= xyz + xyz' + x'yz + x'y'z + \cancel{xyz} + \cancel{x'yz} =$

$= xy(z+z') + x'z(y+y') = xy + x'z$

duality: $(x+y)\cdot(x'+z)(y+z) = (x+y)(x+z')$

Theorem (De Morgan):

$$\left(\sum_{i=1}^{n} x_i\right)' = \prod_{i=1}^{n} x_i' \quad , \quad \left(\prod_{i=1}^{n} x_i\right)' = \sum_{i=1}^{n} x_i'$$

Theorem (Shannon)

$$f(x_1, \dots, x_n) = x_i\, f(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n) + $$
$$+ x_i'\, f(x_1, \dots x_{i-1}, 0, x_{i+1}, \dots, x_n)$$

$$f(x_1, \dots, x_n) = [x_i + f(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n)] +$$
$$+ [x_i' + f(x_1, \dots, x_{i+1}, 1, x_{i+1}, \dots, x_n)]$$

Example: Show $(x+y)[x'(y'+z')]' + x'y' + x'z' = 1$

$$(x+y)[x'(y'+z')]' + x'y' + x'z' \underset{\text{de morgan}}{=}$$
$$= (x+y)[x+(y'+z')'] + x'y' + x'z' \underset{\text{demorgan}}{=}$$

$(xy)' = x'+y'$ De Morgan

$x'+y' = (x'+y')\cdot 1$ (axiom $a=a\cdot 1$)

$= (x'+y')[xy+(xy)']$ (axiom $a+a'=1$)

$= \underbrace{x'xy}_{0} + x'(xy)' + \underbrace{y'xy}_{0} + y'(xy)'$ (distrib.)

$aa'b = (aa')b = 0$ associative commutative

$= x'(xy)'+y'(xy)' = (x'+y')(xy)'$ (distrib.)

$= (x'+y')(xy)'+(xy)(xy)'$ (axiom $aa'=0$)

$= (x'+y'+xy)(xy)'$ (distrib.)

$= (\underbrace{x'+y'+xy}+xy)(xy)'$ (theorem $a+a=a$)

theorem: $x'+y = x'+xy+x'y = x'(1+y)+xy = x'+xy$

$y'+x = y'+xy$

$= (x'+y'+x+y)(xy)' = (1+1)(xy)'$

$= (xy)'$

<u>Homework:</u> Show by induction the general case

Sannon Expansion Theorem

Break into two cases: $x_i=0$ and $x_i=1$

$f(x_1, \ldots x_{i-1}, x_i, x_{i+1}, \ldots, x_n) =$

$= 1 \cdot f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots x_n) +$

$+ 0 \cdot f(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n) =$

$= x_i' \cdot f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) +$

$+ x_i \cdot f(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n)$

Simularly for $x_i=1$

$f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) =$

$= 1 \cdot f(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n) +$

$+ 0 \cdot f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) =$

$= x_i f(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n) +$

$+ x_i' f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)$

Dual Proof Break into two cases

$x_i = 1$

$f(\ldots, x_i, \ldots) =$

$= [0 + f(\ldots, 1, \ldots)][1 + f(\ldots, 0, \ldots)] =$

$= [x_i' + f(\ldots, 1, \ldots)][x_i + f(\ldots, 0, \ldots)]$

$x_i = 0$

$f(\ldots, x_i, \ldots) =$

$= [0 + f(\ldots, 0, \ldots)][1 + f(\ldots, 1, \ldots)] =$

$= [x_i + f(\ldots, 0, \ldots)][x_i' + f(\ldots, 1, \ldots)]$

$$(x+y)[x+yz]+x'y'+x'z'=$$

$$\underbrace{x+\underbrace{xy+xyz}_{\text{absorption}}}_{\text{absorption}}+yz+x'y'+x'z'=$$

$$x+yz+x'y'+x'z'=x+x'(y'+z')+yz=$$

$$\underset{①}{x(y'+z')}+\underset{③}{x(y'+z')'}+\underset{②}{x'(y'+z')}+yz \overset{①+②,\ \text{demorgan}③}{=}$$

$$y'+z'+\underbrace{xyz+yz}_{\text{absorption}}=(yz)'+yz=1$$

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from Boolean to Switching algebra

$$B=\{0,1\}$$

$\{+\}\to$ OR, $\{\cdot\}\to$ AND, $\{'\}\to$ NOT

$+$	$y=0$	$y=1$
$x=0$	0	1
$x=1$	1	1

$\cdot$	$y=0$	$y=1$
$x=0$	0	0
$x=1$	0	1

$'$	
$x=0$	1
$x=1$	0

or alternatively

Truth Table

OR

x	y	x+y
0	0	0
0	1	1
1	0	1
1	1	1

AND

x	y	x·y
0	0	0
0	1	0
1	0	0
1	1	1

NOT

x	x'
0	1
1	0

Proving identities by truth table

$$x + yz = (x+y)(x+z)$$

x	y	z	yz	x+yz	x+y	x+z	(x+y)·(x+z)
0	0	0	0	0	0	0	0
0	0	1	0	0	0	1	0
0	1	0	0	0	1	0	0
0	1	1	1	1	1	1	1
1	0	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	1	0	0	1	1	1	1
1	1	1	1	1	1	1	1

Boolean algebra resembles ordinary algebra but

distribution axiom $a + (b \cdot c) = (a+b) \cdot (a+c)$

doesn't exist in ordinary algebra

complement axiom's do not exist in ordinary algebra

$x = y \Rightarrow x + z = y + z$ in both

$x + z = y + z \not\Rightarrow x = y$ in Boolean

cancelation doesn't exist in Boolean algebra

Boolean Functions

Boolean expressions: $0, 1, \{x_{n-1}, \ldots, x_0\}$

① $0, 1, x_0, \ldots, x_{n-1}$ are expressions

② if E_1 and E_2 are expression, then

$E_1', E_2', E_1 + E_2, E_1 \cdot E_2$ are expression

Expressions are regarded as Boolean (Switching) functions.

- Consider $F(x_{n-1}, \ldots, x_0)$ Boolean function of n variables
- There are 2^n possible assignments (distinct) of $x_{n-1} \ldots, x_0$ with 0 and 1
- Each assignment may result $F=1$ or $F=0$
- Consequently there are 2^{2^n} distinct Boolean functions of n variables.

Different expressions may yield same function

$$\left.\begin{aligned} E_1 &= x_1x_2 + x_1'x_3 + x_2x_3 \\ E_2 &= x_1x_2 + x_1'x_3 \end{aligned}\right\} \text{equivalent by consensus}$$

E_2 can be obtained from E_1 by axioms and laws of Boolean algebra

F', $F+G$ and FG are well defined by NOT, OR and AND, resulting in a new function

<u>Example</u>

x_3	x_2	x_1	F	G	F'	$F+G$	$F \cdot G$
0	0	0	1	1	0	1	1
0	0	1	0	0	1	0	0
0	1	0	1	1	0	1	1
0	1	1	0	0	1	0	0
1	0	0	1	0	0	1	0
1	0	1	0	1	1	1	0
1	1	0	0	1	1	1	0
1	1	1	1	1	0	1	1

2-Variable Functions - Binary logic operations

F_1	0	0	0	1	$F_1 = xy$	and
F_2	0	0	1	0	$F_2 = xy'$	*Inhibit
F_3	0	0	1	1	$F_3 = x$	Transfer
F_4	0	1	0	0	$F_4 = x'y$	*Inhibit
F_5	0	1	0	1	$F_5 = y$	Transfer
F_6	0	1	1	0	$F_6 = xy' + x'y$	Xor $x \oplus y$
F_7	0	1	1	1	$F_7 = x + y$	Or
F_8	1	0	0	0	$F_8 = (x+y)'$	Nor (Not Or)
F_9	1	0	0	1	$F_9 = xy + x'y'$	Equivalence $\odot$
F_{10}	1	0	1	0	$F_{10} = y'$	complement
F_{11}	1	0	1	1	$F_{11} = x + y'$	*y implies x
F_{12}	1	1	0	0	$F_{12} = x'$	complement
F_{13}	1	1	0	1	$F_{13} = x' + y$	x implies y
F_{14}	1	1	1	0	$F_{14} = (xy)'$	Nand (not and)
F_{15}	1	1	1	1	$F_{15} = 1$	constant

* hardly in use		1	1	0	0	x
		1	0	1	0	y
constant	$F_0 = 0$	0	0	0	0	F_0
and	$F_1 = xy$	1	0	0	0	F_1
Inhibit	$F_2 = xy'$	0	1	0	0	F_2
Transfer	$F_3 = x$	1	1	0	0	F_3
:Inhibit	$F_4 = x'y$	0	0	1	0	F_4
Transfer	$F_5 = y$	1	0	1	0	F_5
Xor $x \oplus y$	$F_6 = xy' + x'y$	0	1	1	0	F_6
Or	$F_7 = x + y$	1	1	1	0	F_7
Nor (Not Or)	$F_8 = (x+y)'$	0	0	0	1	F_8
quivalence $\odot$	$F_9 = xy + x'y'$	1	0	0	1	F_9
complement	$F_{10} = y'$	0	1	0	1	F_{10}
y implies x	$F_{11} = x + y'$	1	1	0	1	F_{11}
complement	$F_{12} = x'$	0	0	1	1	F_{12}
x implies y	$F_{13} = x' + y$	1	0	1	1	F_{13}

Extension to Multiple Variables

- OR and AND are commutative and associative, hence can be extended to any number of variables.
- XOR is commutative and associative so can be extended
- 2-variable equivalence is complement of XOR, hence associative and commutative
- However, 3-variable $\odot$ is not complement of $\oplus$ but rather equal.

$$x \oplus y \oplus z = (x \oplus y) \oplus z = (x'y + xy')z' +$$

$$(x'y + xy')'z \overset{\text{De-Morgan}}{=} x'yz' + xy'z' + (x+y')(x'+y)z =$$

$$= xyz + xy'z' + x'yz' + x'y'z$$

x	0	0	0	0	1	1	1	1
y	0	0	1	1	0	0	1	1
z	0	1	0	1	0	1	0	1
	0	1	1	0	1	0	0	1

$x \odot y \odot z = (x'y' + xy)z + (x'y' + xy)'z' =$

$x'y'z + xyz + (x+y)\cdot(x'+y')z' =$

$xyz + x'y'z + xy'z' + x'yz'$

We obtain same truth table

In general for even # variables $\oplus$ and $\odot$ are complementary, while for odd # they are similar

$\oplus$ XOR is odd function: for odd # of 1's it equals 1

Proof: for $n=2$ trivial. Assume it is true for $k-1$

$x_0 \oplus x_1 \oplus \cdots \oplus x_{k-1} \oplus x_k =$

$(x_0 \oplus x_1 \oplus \cdots \oplus x_{k-1}) \oplus x_k = \omega \oplus x_k =$

$= \omega x_k' + \omega' x_k$

let # 1's be odd (even) — $x_k = 0$ / $x_k = 1$

$\odot$ Equivalence is even: for even # 0's it equals 1

Proof: By induction. $n=2$ trivial

$$x_0 \odot \cdots \odot x_{k-1} \odot x_k = (x_0 \odot \cdots \odot x_{k-1}) \odot x_k =$$

$$= W \odot x_k = W x_k + W' x_k'$$

Let # 0's be even (odd) $\begin{cases} x_k = 0 \\ x_k = 1 \end{cases}$

NOR and NAND are commutative but not associative

Proof:

$$((x+y)' + z)' \underset{\substack{\uparrow \\ \text{DeMorgan}}}{=} (x+y) \cdot z' = xz' + yz'$$

$$(x + (y+z)')' = x'(y+z) = x'y + x'z$$

Similar proof for NAND

Redefinition of multi-variable NOR and NAND

$$(x_1 + x_2 + \cdots + x_n)'$$

$$(x_1 \cdot x_2 \cdot \cdots \cdot x_n)'$$

Functional Completeness

A set of operations is functionally complete iff every Boolean function can be expressed in terms of these operations

$\{+, \cdot, '\}$ is complete because this is how it was defined

$\{\cdot, '\}$ is complete since

$$x + y \underset{\text{De-Morgan}}{=} (x'y')'$$

$\{+, '\}$ is complete since

$$x \cdot y \underset{\text{De-Morgan}}{=} (x' + y')'$$

$\{NOR\}$ is complete since

$$\underbrace{x'}_{NOT} = x'x' = (\underbrace{x+x}_{NOR})'$$

$$\underbrace{(x+y)}_{OR} = (x+y)\cdot(x+y) = \underbrace{((x+y)'+(x+y)')'}_{NOR}$$

<u>Example</u>: The set $\{f, 1\}$

$$f(x,y,z) = x'y + x'z + y'z'$$

is complete

$$f(x,x,x) = x'x' + x'x' + x'x' = x'x' = \underbrace{x'}_{NOT}$$

$$f(f(x,x,x), f(y,y,y), 1) = f(x',y',1) =$$

$$xy + x\cdot 0 + y\cdot 0 = \underbrace{xy}_{AND}$$

Complements

Use either De Morgan or dual and then complement every variable

$$f = xy'z + x'yz'$$

$$f' = (xy'z + x'yz')' = (xy'z)'\cdot(x'yz')' =$$

$$= (x'+y+z')(x+y'+z)$$

$f = xy'z + x'yz' \rightarrow (x+y'+z)(x'+y+z') = f^d$

now complement every variable

$f' = (x'+y+z')(x+y'+z)$

Boolean Canonical Forms: Derivation of Boolean expressions from truth table

$xy, x'y, xy', x'y'$ - AND - product terms

$x+y, x'+y, x+y', x'+y'$ - OR - sum terms

any of x, x' is called literal (same y, y')

$x_1^* \cdots x_n^*$ - product of n distinct literals is called minterm (minimum)

$x_1^* + \cdots + x_n^*$ - sum of n distinct literals is called maxterm (maximum)

* We seek for combination of $\underbrace{\text{literals}}^{\text{values of}}$ which defines uniquely minterms and maxterms

* Since minterm is a product of literals, it is unique when all the literals are 1

Maxterm is unique when all literals are 0

Decimal	x	y	z	minterm	Maxterm
0	0	0	0	$x'y'z'$ (m_0)	$x+y+z$ $(M_0$
1	0	0	1	$x'y'z$ (m_1)	$x+y+z'$ $(M_1$
2	0	1	0	$x'yz'$ (m_2)	$x+y'+z$ $(M_2$
3	0	1	1	$x'yz$ (m_3)	$x+y'+z'$ $(M_3$
4	1	0	0	$xy'z'$ (m_4)	$x'+y+z$ $(M_4$
5	1	0	1	$xy'z$ (m_5)	$x'+y+z'$ $(M_5$
6	1	1	0	xyz' (m_6)	$x'+y'+z$ $(M_6$
7	1	1	1	xyz (m_7)	$x'+y'+z'$ $(M_7$

Example:

m_{13} - minterm $(13)_{10} = (01101)_2 : x_4' x_3 x_2 x_1' x_0$

M_{17} - Maxterm $(17)_{10} = (10001)_2 : x_4' + x_3 + x_2 + x_1 + x_0$

$$m_i m_j = \begin{cases} 0 & i \neq j \\ m_i & i = j \end{cases}, \quad M_i + M_j = \begin{cases} 1 & i \neq j \\ M_i & i = j \end{cases}$$

$m_i = M_i'$, $M_i = m_i'$ by De-Morgan

How to derive Boolean expression from truth table?

- Summing (OR ing) minterms for which the function is 1 in table
- Multiplying (ANDing) maxterms for which the function is 0 in table

Example

Decimal	x_2	x_1	x_0	f
0	0	0	0	0
1	0	0	1	1
2	0	1	0	0
3	0	1	1	0
4	1	0	0	1
5	1	0	1	1
6	1	1	0	1
7	1	1	1	1

$$f(x_2, x_1, x_0) = m_1 + m_4 + m_5 + m_6 + m_7 =$$

$$x_2' x_1' x_0 + x_2 x_1' x_0' + x_2 x_1' x_0 + x_2 x_1 x_0' + x_2 x_1 x_0 =$$

$$\sum(1, 4, 5, 6, 7)$$

This is called Canonical Sum Of Products SOP

$$f(x_2, x_1, x_0) = M_0 M_2 M_3 =$$

$$(x_2 + x_1 + x_0)(x_2 + x_1' + x_0)(x_2 + x_1' + x_0') =$$

$$= \prod(0, 2, 3)$$

This is called Canonical Product Of Sums POS

Theorem: Every Boolean function $f(x_{n-1}, \dots, x_0)$ can be written as

$$f(x_{n-1}, \dots, x_0) = \sum_{k=0}^{2^n-1} \alpha_k m_k \quad \text{Canonical SOP}$$

or

$$f(x_{n-1}, \dots, x_0) = \prod_{k=0}^{2^n-1} (\beta_k + M_k) \quad \text{Canonical POS}$$

0/1 decimal

$$\alpha_k = \beta_k = f(k) = f(k_{n-1}, \dots, k_0)$$

Example

How to obtain Canonical SOP for all $2^{2^3} = 2^8 = 256$ functions of 3 variables?

$$f(x,y,z) \stackrel{\text{shannon}}{=} x f(1,y,z) + x' f(0,y,z) \stackrel{\text{shannon}}{=}$$

$$x[y f(1,1,z) + y' f(1,0,z)] + x'[y f(0,1,z) + y' f(0,0,z)] =$$

$$= \ldots \stackrel{\text{shannon}}{=}$$

$$f(0,0,0)x'y'z' + f(0,0,1)x'y'z + f(0,1,0)x'yz' +$$
$$f(0,1,1)x'yz + f(1,0,0)xy'z' + f(1,0,1)xy'z +$$
$$f(1,1,0)xyz' + f(1,1,1)xyz$$

Each of $f(\cdot,\cdot,\cdot)$ is either 0 or 1, depending on f.

Similarly $f(x,y,z) \stackrel{\text{shannon-dual}}{=} [x + f(0,y,z)][x' + f(1,y,z)] =$

$$= [x + [y + f(0,0,z)][y' + f(0,1,z)]] \cdot$$
$$[x' + [y + f(1,0,z)][y' + f(1,1,z)]] = \ldots$$

Proof of Theorem

By Induction. Assume that hypothesis exists for $n-1$ variables $x_{n-2}, \ldots, x_0$. Then, by Shannon theorem

$$f(x_{n-1}, x_{n-2}, \ldots, x_0) =$$

$$x_{n-1} f(1, x_{n-2}, \ldots, x_0) + x_{n-1}' f(0, x_{n-2}, \ldots, x_0).$$

By induction

$$\left.\begin{array}{l} f(1, x_{n-2}, \ldots, x_0) = \sum_{k=0}^{2^{n-2}-1} \alpha_k m_k \\ \qquad \alpha_k = f(1, \underbrace{k_{n-2}, \ldots, k_0}_{\text{or decimal } k}) \\ \text{similarily} \\ f(0, x_{n-2}, \ldots, x_0) = \sum_{k=0}^{2^{n-2}-1} \alpha_k m_k \\ \qquad \alpha_k = f(0, \underbrace{k_{n-2}, \ldots, k_0}_{\text{or decimal } k}) \end{array}\right\} *$$

$$* = x_{n-1} \sum_{k=0}^{2^{n-2}-1} \alpha_k m_k + x_{n-1}' \sum_{k=0}^{2^{n-2}-1} \alpha_k m_k$$

Relations between SOP and POS canonical forms

$$SOP \overset{\text{De Morgan}}{\Longleftrightarrow} POS$$

<u>Example</u>

$$f(x_2, x_1, x_0) = \sum(1,4,5,6,7)$$

$$f'(x_2, x_1, x_0) = \sum(0,2,3)$$

$$f = (f')' = \left[\sum(0,2,3)\right]' = m_0' m_2' m_3' = M_0 M_2 M_3 =$$

$$= \Pi(0,2,3)$$

Complements and dual of canonical forms

Let $f = \sum_{i=0}^{2^n-1} \alpha_i m_i$ - canonical SOP

then $f' = \left(\sum_{i=0}^{2^n-1} \alpha_i m_i\right)' \underset{\text{De Morgan}}{=} \prod_{i=0}^{2^n-1} (\alpha_i' + m_i') \underset{m_i' = M_i}{=} \prod_{i=0}^{2^n-1} (\alpha_i' + M_i)$

Let $f = \prod_{i=0}^{2^n-1} (\beta_i + M_i)$

$$f' = \left(\prod_{i=0}^{2^n-1} (\beta_i + M_i)\right)' = \sum_{i=0}^{2^n-1} \beta_i' M_i' = \sum_{i=0}^{2^n-1} \beta_i' m_i$$

<u>Example</u> $f = \Sigma(1,2,3,4) = \Pi(0,5,6,7)$

$\bar{\alpha} = (0,1,1,1,1,0,0,0)$, $\bar{\beta} = (0,1,1,1,1,0,0,0)$

$f' = (\Sigma(1,2,3,4))' =$

$(1+M_0)(0+M_1)(0+M_2)(0+M_3)(0+M_4)(1+M_5)(1+M_6)(1+M_7)$

$\Pi(1,2,3,4)$

$f' = (\Pi(0,5,6,7))' =$

$= 1\cdot m_0 + 0\cdot m_1 + 0\cdot m_2 + 0\cdot m_3 + 0\cdot m_4 + 1\cdot m_5 + 1\cdot m_6 + 1\cdot m_7 =$

$\Sigma(0,5,6,7)$

Duals

$$f^d = \left(\sum_{i=0}^{2^n-1} \alpha_i m_i\right)^d = \prod_{i=0}^{2^n-1} (\alpha_i^d + m_i^d) = \prod_{i=0}^{2^n-1} (\alpha_i^d + M_{2^n-1-i}$$

$$f^d = \left[\prod_{i=0}^{2^n-1} (\beta_i + M_i)\right]^d = \sum_{i=0}^{2^n-1} \beta_i^d M_i^d = \sum_{i=0}^{2^n-1} \beta_i\, m_{2^n-1-i}$$

Example

$$f^d = [\Sigma(1,2,3,4)]^d = (m_1 + m_2 + m_3 + m_4)^d =$$

$$= m_1^d m_2^d m_3^d m_4^d = M_6 M_5 M_4 M_3$$

$$f^d = [\Pi(0,5,6,7)]^d = (M_0 M_5 M_6 M_7)^d =$$

$$= M_0^d + M_5^d + M_6^d + M_7^d = m_7 + m_2 + m_1 + m_0$$

Finding the expression of dual means finding the expression for which the function =0 rather than function =1. This is not complement

Non Canonical forms - Simplified forms

Simplified or minimized forms are **essential** for efficient hardware.

Conversion to canonical - may be required for simplification.

Example: Convert $f(x,y,z)=x'y+z+xy'$ to canonical SOP

$$x'y+z+xy' \overset{\text{what is missing in every product}}{=} x'y\underbrace{(z+z')}_{1} + \underbrace{(x+x')}_{1}\underbrace{(y+y')}_{1}z + xy'(z+z') = x'yz + \overset{2}{x'yz'} + xyz + x'y'z + xy'z + \overset{4}{x}y'z' = \Sigma(1,2,3,4,5,7)$$

Further conversion to canonical POS - use maxterms where f is zero

$$f(x,y,z) = \Pi(0,6) = (x+y+z)(x'+y'+z)$$

Example: Convert $f(x,y,z)=x(y'+z)$ to canonical POS

$$x = x+yy' \overset{\text{distribution}}{=} (x+y)(x+y') \overset{\text{distribution}}{=} (x+y+zz')(x+y'+zz')$$

$$\overset{\text{distribution}}{=} [(x+y+z)(x+y+z')][(x+y'+z)(x+y'+z')]$$

$$y'+z = xx'+y'+z \underset{\text{distribution}}{=} (x+y'+z)(x'+y'+z)$$

$$f(x,y,z) = M_0 \cdot M_1 \cdot M_2 \cdot M_3 \cdot M_2 \cdot M_6 = \Pi(0,1,2,3,6)$$