3.3 Matchings and Factors: Matchings in General Graphs

Overview of Matchings in General Graphs

- Section 3.1 used **Hall's Condition** to characterize which X,Y-bigraphs have an X-saturating matching. We also saw the maximum size matching equals the minimum size vertex cover for bipartite graphs.
- Section 3.2 extended this to bipartite matchings and vertex covers for weighted edges, via the **Hungarian Algorithm** which finds a maximum weight matching with weight equal to a minimum weight cover.

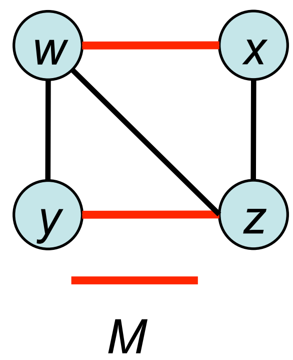
- Now in Section 3.3 **Tutte's Condition** characterizes all graphs having a perfect matching, and more generally the size of a maximum matching in any graph.

Factors

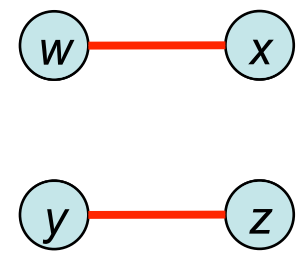
<u>3.3.1 Definition</u> A **factor** of a graph G is a spanning subgraph of G. A k-factor is a spanning k-regular subgraph. An **odd component** of a graph is a component of odd order; the number of odd components of H is $o(H)$.

Perfect matching M

$\{wx, yz\}$



1-factor



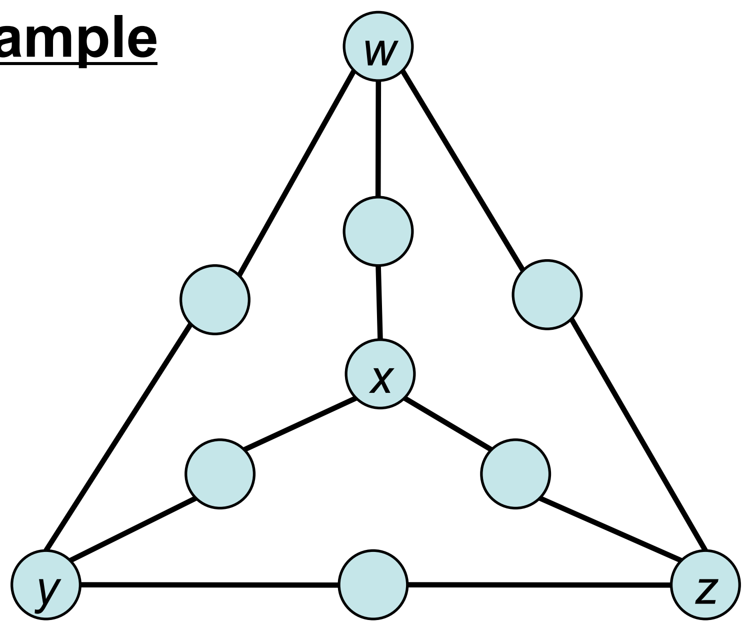
Perfect matchings precisely correspond to 1-factors by including the vertices of the graph with the edges of the matching.

Tutte’s Condition

William Tutte (1917-2002) (see Wikipedia) proved in 1947 that any graph satisfying the following condition has a 1-factor:

For all $S \subseteq V(G)$, $o(G-S) \leq |S|$. (Tutte’s Condition)

Example



Is there a 1-factor?

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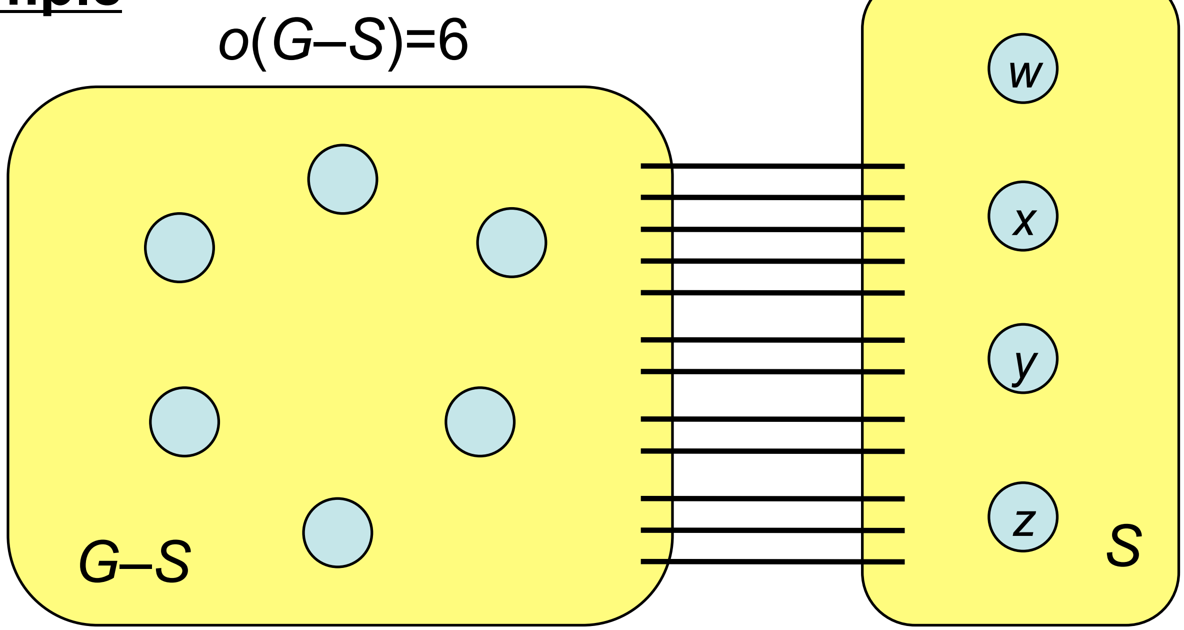

No 1-factor: $o(G-S)=6 \geq |S| = 4$

Tutte's 1-Factor Theorem

3.3.3 Theorem . (Tutte [1947]) A graph G has a 1-factor if and only if $o(G–S) \leq |S|$ for every $S \subseteq V(G)$.

Proof

(=>) If G has a 1-factor, the odd components of $G–S$ must have at least one vertex each matched to vertices of S.

(<= by contradiction)

Assume Tutte's condition holds and assume to the contrary G has no 1-factor. Without loss of generality we may assume:

1. G is simple and has no 1-factor.
2. $G+e$ has a 1-factor for every $e \notin E(G)$.
3. G satisfies Tutte's condition.

Next we justify these 3 assumptions.

Tutte's 1-Factor Theorem

Assume Tutte's condition holds and assume to the contrary G has no 1-factor. Without loss of generality we may assume:

1. G is simple and has no 1-factor:

Justification

If G has no 1-factor, neither does any simple subgraph of G.

Replace G with a simple graph by removing all loops and all but one edge incident to any given pair of vertices.

Note that for every $S \subseteq V(G)$ $o(G-S)$ does not change under this transformation!

Tutte's 1-Factor Theorem

Assume Tutte's condition holds and assume to the contrary G has no 1-factor. Without loss of generality we may assume:

1. G is simple and has no 1-factor.
2. $G+e$ has a 1-factor for every $e \notin E(G)$:

Justification

We have a simple graph G with no 1-factor.

There must exist a supergraph of G on vertices $V(G)$ that is edge-maximal with respect to having no 1-factor.

For some $e \notin E(G)$, if $G+e$ has no 1-factor, we simply replace G by $G+e$ and repeat.

Now we have a simple G with no 1-factor, but $G+e$ has a 1-factor for every $e \notin E(G)$.

Tutte's 1-Factor Theorem

Assume Tutte's condition holds and assume to the contrary G has no 1-factor. Without loss of generality we may assume:

1. G is simple and has no 1-factor.
2. $G+e$ has a 1-factor for every $e \notin E(G)$.
3. G satisfies Tutte's condition:

Justification

We have a simple graph G with no 1-factor but $G+e$ has a 1-factor for every $e \notin E(G)$.

Now let $S \subseteq V(G)$ be arbitrary. Step 1 did not change $o(G–S)$.

Adding edges to G to get to Step 2 did not increase $o(G–S)$:

$G–S$ contains such an added edge only when both endpoints are within $G–S$, and its number of odd components can only stay the same or decrease.

(joining components: odd-even -> odd, odd-odd -> even, even-even -> even)

Tutte's 1-Factor Theorem

<u>3.3.3 Theorem . (Tutte [1947])</u> A graph G has a 1-factor if and only if $o(G–S) \leq |S|$ for every $S \subseteq V(G)$.

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(=>) If G has a 1-factor, the odd components of $G–S$ must have at least one vertex each matched to vertices of S.

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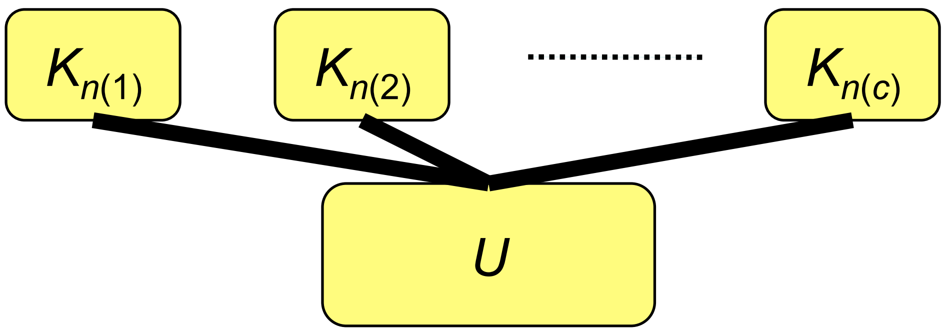
We continue proof of (<=) with G satisfying Properties 1-3.

Tutte's 1-Factor Theorem

Proof of (<=)

Define $U = \{v \in V(G) : d(v) = n(G)-1\}$.

Case 1 The components of $G-U$ are complete graphs.



If $n(i)$ is even, then $K_{n(i)}$ has a perfect matching.

Define $t = |\{i : n(i) \text{ is odd}\}|$. For odd $n(i)$, $K_{n(i)}$ has a matching saturating all but 1 vertex. So far we can match all but t vertices in $G-U$.

By Tutte's Condition, $t \leq |U|$. These t vertices are all adjacent to all vertices of U, and can be matched to t vertices of U.

Tutte's 1-Factor Theorem

Proof of (<=)

Define $U = \{v \in V(G) : d(v) = n(G)-1\}$.

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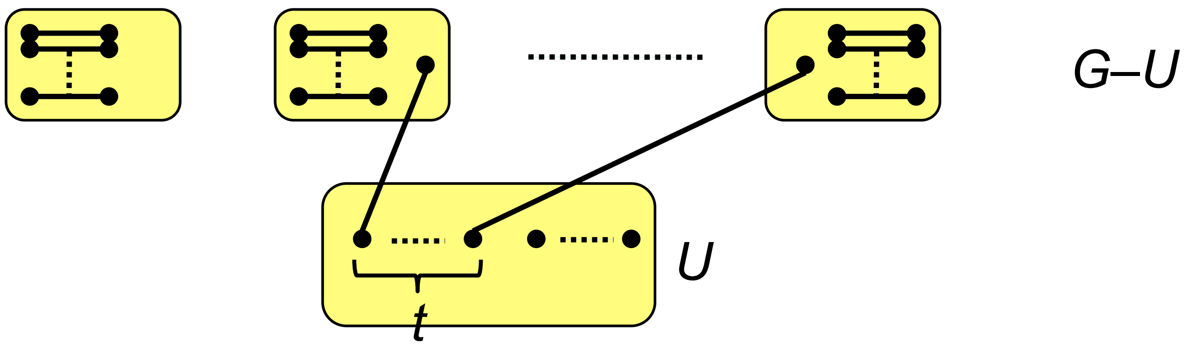


The t last vertices of the odd complete components of $G-U$ are matched to t vertices of U.

Claim $|U|$ is even, and so G has a 1-factor after all.

It is easy to see that $|U|-t$ and n(G) have the same parity.

Also, $n(G)$ must be even. Otherwise $|\varnothing| = 0 < 1 \leq o(G-\varnothing) = o(G)$.

Complete the 1-factor by pairing the remaining $|U|-t$ vertices of U.

Tutte's 1-Factor Theorem

Proof of (<=)

Define $U = \{v \in V(G) : d(v) = n(G)-1\}$.

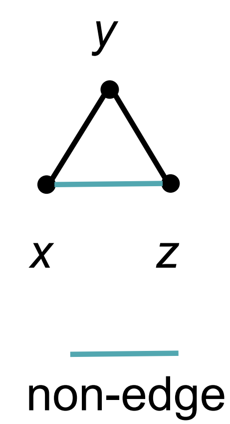
Case 2 Some component of $G-U$ is not complete.

Therefore $G-U$ has vertices $x,y,z \in V(G)-U$ with:

(1) $d_{G-U}(x,z) = 2$ and $xz \notin E(G)$;

(2) xy, $yz \in E(G-U)$; and

(3) $d(y) < n(G) - 1$.



The existence of y follows from the distance-2 condition on x,y.

This distance is with respect to $G-U$, so $y \notin U$.

By definition of U, y is adjacent to all vertices of U but not all vertices of G. Therefore $G-U$ has a vertex w with:

(4) $yw \notin E(G)$.

Tutte's 1-Factor Theorem

Proof of (<=)

Define $U = \{v \in V(G) : d(v) = n(G)-1\}$.

Case 2 Some component of $G-U$ is not complete.

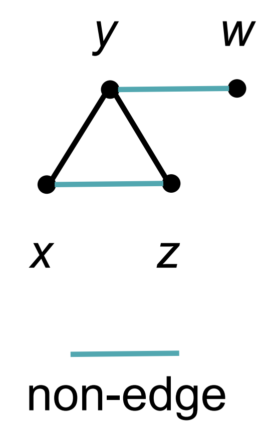
Therefore $G-U$ has vertices $w,x,y,z \in V(G)-U$ with:

(1) $d_{G-U}(x,z) = 2$ and $xz \notin E(G)$;

(2) xy, $yz \in E(G-U)$; and

(3) $d(y) < n(G) - 1$;

(4) $yw \notin E(G)$.



By assumption that adding any edge to G yields a 1-factor:

(5) $G+xz$ has a 1-factor – call it M_1;

(6) $G+yw$ has a 1-factor – call it M_2.

Tutte's 1-Factor Theorem

<u>Proof of (<=)</u>

We show it $M_1 \cup M_2 \cup \{xy, yz\}$ contains a 1-factor avoiding $\{xz, yw\}$.

Define $F = M_1 \Delta M_2$, which contains both xz and yw.

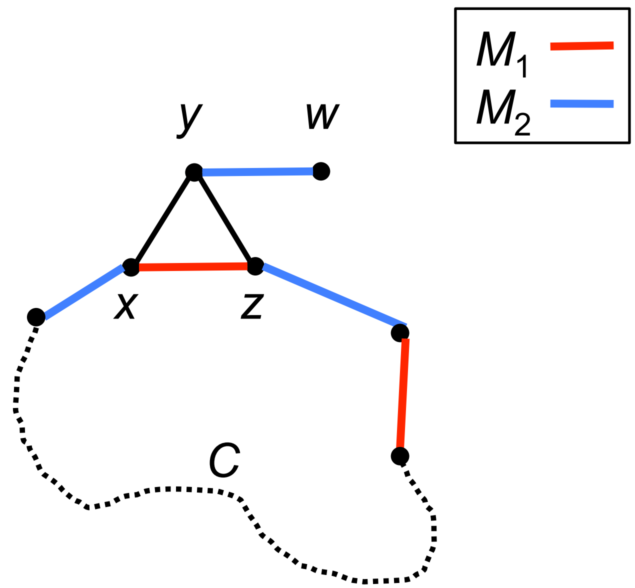
<u>Fact</u> The components of F are even cycles and isolated vertices, because the degree of a vertex in $M_1 \Delta M_2$ is 0 or 2.

Define C to be the cycle of F containing xz.

<u>Case A</u> C does not contain yw.

Define a 1-factor on all of G by selecting:

- The edges of M_2 on C
- The edges of M_1 everywhere not on C.



Tutte’s 1-Factor Theorem

<u>Proof of (<=)</u>

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Define $F = M_1 \Delta M_2$, which contains both xz and yw.

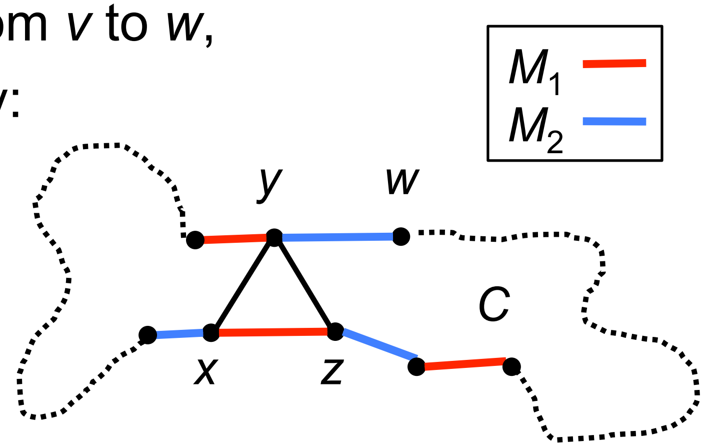
Define C to be the cycle of F containing xz.

<u>Case B</u> C contains yw.

When traveling around C in the direction from v to w,

if z is reached first, define a 1-factor of G by:

- Selecting M_1 between v and z on this side;
- Selecting the edge yz;
- Selecting M_2 on the other side of C; and
- Selecting exactly one of M_1, M_2 off of C.



If x is encountered first instead of z,

replace yz with xz above.