**<u>שאלה 1 (60 נק')</u>**

Prove that for every **<u>directed</u>** graph $G(V,E)$ there is an independent set $S \subseteq V$ such that $\forall v \in V$ there is $d(v,S) \leq 2$.

In other words, every $v \in V$ has some $s(v) \in S$ such that the distance (length of shortest **<u>directed</u>** path) from $s(v)$ to $v$ has at most two arcs.

**Hints**:

1. Use induction on $V$.
2. Let $x \in V$ and $T = \{ t \in V \mid \vec{e}(x,t) \in E\}$
3. Consider $G'(V',E')$, where $V' = V \backslash \{T \cup \{x\}\}$, and apply induction.

**<u>Solution</u>**

Let $x \in V$ and $T = \{ t \in V \mid \vec{e}(x,t) \in E\}$.

Consider $G'(V',E')$, where $V' = V \backslash \{T \cup \{x\}\}$.

Assume by induction that there is an independent vertex set $S' \subseteq V'$ such that $\forall v \in V' \backslash S'$ there is $d(v,S') \leq 2$.

There are two cases:

1. If $S' \cup \{x\}$ is independent set $S = S' \cup \{x\}$.
   Then by induction all the vertices of $V' \backslash S'$ have distance at most 2 from $S$, and all $v \in T$ have distance 1 from $S$ (in fact from $x$).
2. $S' \cup \{x\}$ is not independent, i.e., $\exists z \in S'$ adjacent to $x$. Since $z \notin T$ there is $\vec{e}(z,x) \in E$.
   Now set $S = S'$. If $y \in V' \backslash S'$ then by induction hypothesis its distance from $S'$ is at most 2.
   If $y \in T$ it can be reached from $S'$ by $\vec{e}(z,x)$ and $\vec{e}(x,y)$. ■

**<u>שאלה 2 (60 נק')</u>**

Prove that a tournament $T(V, E)$ is **strongly connected** if and only if it has a Hamiltonian cycle.

**Reminder**: A tournament $T$ is a directed complete graph (all edges are directed).

**Reminder**: A Hamiltonian cycle traverses all the vertices of $T$ (direction preserved along the cycle).

**Hints**:

1. For "only if" assume there is maximum cycle $C: (v_1, v_2, \dots, v_k)$ but not Hamiltonian.
2. Let $x \notin C$, and $\vec{e}(v_1, x) \in E$. Conclude the directions of all the arcs involving $x$ and $v_i \in C$.
3. Let $X$ be all varices as $x$ in 2. Prove that there is a vertex neither in $X$ nor in $C$ and conclude the consequences on $C$.

**Solution**

The proof of "if" is trivial. If the tournament $T(V, E)$ has a Hamiltonian cycle then all the vertices are connected along the cycle.

For "only if", suppose that $T$ is strongly connected. $T$ therefore has cycles.

Let $C: (v_1, v_2, \dots, v_k)$ be maximum cycle but not Hamiltonian.

Let $x \notin C$, and $\vec{e}(v_1, x) \in E$. If there is $\vec{e}(x, v_2) \in E$, $C': (v_1, x, v_2, \dots, v_k)$ is a longer cycle than $C$, a contradiction. Hence $\vec{e}(v_2, x) \in E$. And similarly $\vec{e}(v_i, x) \in E$ for $i = 1,2, \dots, k$ .

Let $X \subset V$ be all such vertices as above $x$.

Since $T$ is strongly connected $X$ must have an outgoing arc so its vertices could have directed paths to $V \backslash X$.

Let $\vec{e}(x, z) \in E$ be such arc, where $x \in X$ and $z \notin X$. Then $z \notin C$ either. Since $z \notin X$, there is $\vec{e}(z, v_j) \in E$. There is also $\vec{e}(v_{j-1}, x) \in E$. Consequently $\vec{e}(v_{j-1}, x)$, $\vec{e}(x, z)$ and $\vec{e}(z, v_j)$ yield longer cycle $(v_1, v_2, \dots, v_{j-1}, x, z, v_j, \dots, v_k)$, so $C$ could not be maximum cycle. ■