<u>שאלה 1 (60 נק')</u>

Let $G(V,E) = [X,Y]$ be simple bipartite, having perfect matching and $d(x) \geq k, \forall x \in X$, and $d(y) \geq k, \forall y \in Y$. Prove that there are at lease $k!$ distinct perfect matchings in G.

Hints:

1. Use induction on both $|V|$ and k.
2. Fix $x \in X$ and $\forall y \in \Gamma(x)$ select $e(x,y)$ and then apply induction.

<u>Proof</u>:

By induction on both $|V|$ and k.

For induction base consider $n = 1$ and $k = 1$. This is a single-edge graph having unique perfect matching and $k! = 1! = 1$ indeed.

Because there is a perfect matching in G, there is $|X| = |Y| = n$.

Let $x \in X$ and $y \in Y$ such that $e(x,y) \in E$.

Consider $G'(V',E') = [X',Y']$, where $X' = X \backslash x$ and $Y' = Y \backslash y$.

There is $|X'| = |Y'| = n - 1$ and $d(x) \geq k - 1$, $x \in X'$, and $d(y) \geq k - 1$, $y \in Y'$.

By induction G' has at least $(k-1)!$ distinct perfect matchings, corresponding to $(k-1)!$ Distinct perfect matching in G, each possessing $e(x,y)$.

Since $d(x) \geq k$ we can select $e(x,y)$, $y \in \Gamma(x)$, in at least k distinct ways. Repeat the above argument for each $e(x,y)$, thus yielding at least $k(k-1)! = k!$ distinct perfect matchings in G. ■

שאלה 2 (60 נק')

Let $G(V,E) \in \mathbf{G}_{n,p}$, a simple graph in the space of all n-vertex graphs with probability $0 < p(n) < 1$ defined on their edges.

Let $p(n) = o(1/n)$, i.e. $pn \underset{n\to\infty}{\to} 0$, and Let $3 \le k$ a fixed integer independent of n.

Prove that G is almost surly C_i-free for all $3 \le i \le k$.

Proof:

Let X^i define the total number of distinct C_i in G.

A C_i is obtained by fixing i vertices and then defining their cyclic order. Note that different orders may yield same cycle (e.g. cyclic shift or reversed order), hence $i!$ bounds the number of underlying distinct cycles.

Let $X_S(C_i)$ be an indicator random variable of whether an i-vertex set S yields the cycle C_i, namely $X_S(C_i) = \begin{cases} 1 & S \text{ yields } C_i \\ 0 & \text{otherwise} \end{cases}$.

Hence $X^i = \sum_S \sum_{C_i} \{X_S(C_i) : S \subseteq V,\ |S| = i\}$.

Let X be the number of distinct C_i in G over all $3 \le i \le k$.

Then $X = \sum_{i=3}^{k} X^i$.

$\mathrm{E}[X_S(C_i)] = 1 \times p^i + 0 \times \left(1 - p^i\right) = p^i$.

By expectation linearity,

$$\mathrm{E}[X] = \sum_{i=3}^{k} \mathrm{E}\left[X^i\right] = \sum_{i=3}^{k} \sum \mathrm{E}[X_S(C_i)] < \sum_{i=3}^{k} \binom{n}{i} i!\, p^i < \sum_{i=3}^{k} \frac{n^i}{i!} i!\, p^i < \sum_{i=3}^{k} (pn)^i < k(pn)^i \underset{n\to\infty}{\rightarrow} 0\,. \blacksquare$$