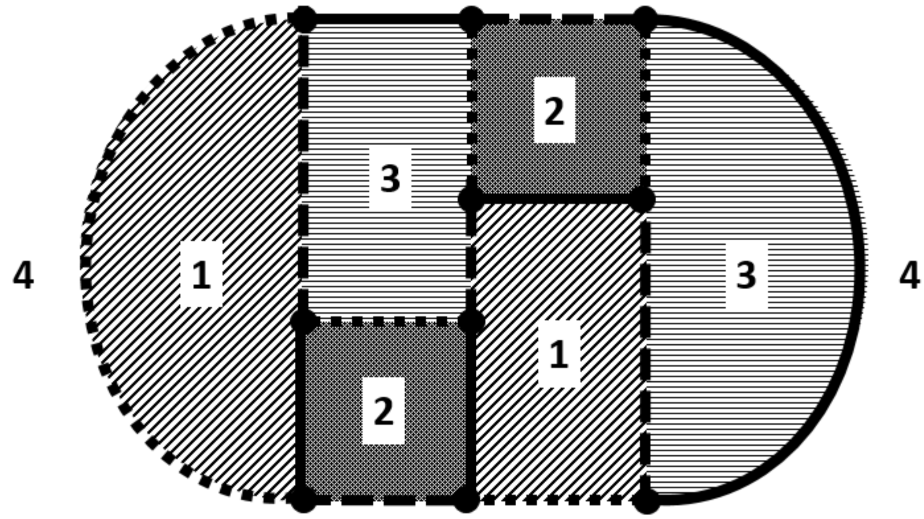
Problem 1 (60 Pts.)

Given a planar 3-regular 2-connected graph. Prove that its faces are 4-colorable if and only if its edges are 3-colorable.

The graph below is an example where the outer face is white (color 4).



Hints for face 4-colorable ⇒ edge 3-colorable:

1. Find all the possible color adjacencies between two faces.
2. Consider what edges between two faces can assume same color.

Hints for edge 3-colorable ⇒ face 4-colorable:

1. Define a subgraph G_{1-2} by deleting all the edges of color 3 and conclude on its planarity and its face coloring.
2. Define similarly G_{1-3} by deleting all the edges of color 2.
3. Consider the merging (overlapping) G_{1-2} and G_{1-3} and conclude on G coloring.

Solution

Face 4-colorabie ⇒ edge 3-colorable:

1. Notice that an edge is defined between two faces.
2. 4 face colors define 6 distinct face adjacencies (1-2, 1-3, 1-4, 2-3, 2-4, 3-4).
3. A specific face color (e.g. 1) defines 3 edge colors (1-2, 1-3, 1-4).
4. The face adjacency 3-4 edge can use the same color as 1-2 face adjacency edge. Otherwise all the four faces meet at a vertex forming a 4-degree vertex.
5. Similarly for 2-4 and 1-3.
6. Similarly for 2-3 and 1-4.
7. Hence all the 6 edge colors are defined properly.

Edge 3-colorable ⇒ face 4-colorable:

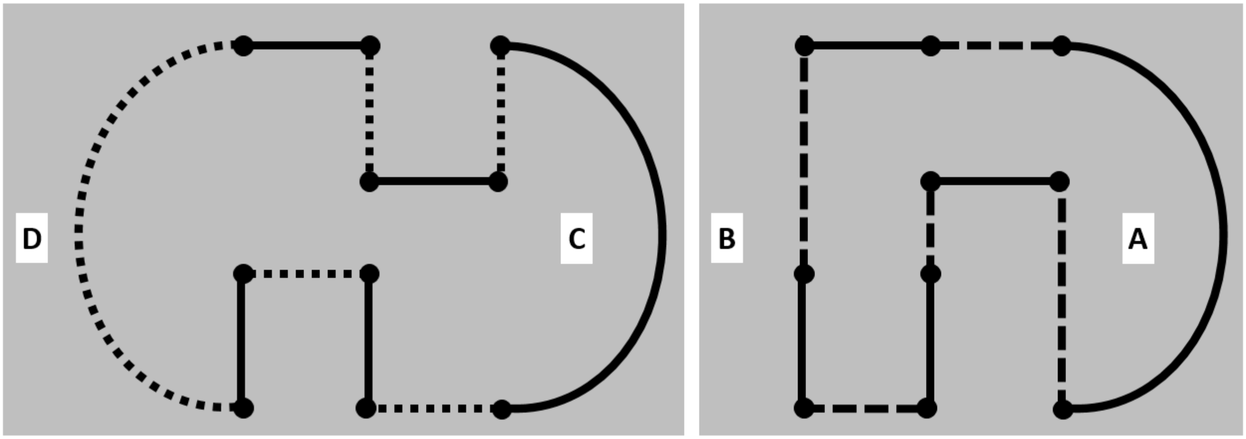
Define a subgraph G_{1-2} by deleting all the edges of color 3. The degree of all the edges in G_{1-2} is 2, hence it comprises only disjoint cycles.

G_{1-2} is hence planar and face 2-colorable. Assign color A to the interior of the cycles and color B to the outer unbounded face.

Similarly, define a subgraph G_{1-3} by deleting all the edges of color 2. The degree of all the edges in G_{1-3} is 2, hence it comprises only disjoint cycles.

G_{1-3} is hence planar and face 2-colorable. Assign color C to the interior of the cycles and color D to the outer unbounded face.

Embed G_{1-2} and G_{1-3} by overlapping each other. The outcome is a planar drawing of G where the faces are the intersection of the colors A-C, A-D, B-C, B-D, hence 4-colorable properly.



Problem 2 (60 Pts.)

Let $G = [X, Y]$ be a bipartite graph, $S \subseteq X$ and $\Gamma(S) \subseteq Y$ S's neighbors. Let

$$\delta = \max_{S \subseteq X} \{|S| - \Gamma(S)\}.$$

Prove that the maximum matching $\alpha'(G)$ satisfies

$$\alpha'(G) = |X| - \delta.$$

Hints:

1. Show that minimum vertex cover $\beta(G) \leq |X| - \delta$ by using the cover $\Gamma(S) \cup (X - S)$, where S yields δ.
2. Use a minimum vertex cover Z and $S = X - Z$ to show $|\Gamma(S)| \leq |Z \cap Y|$.
3. Obtain the cover on Y from Z by removing its vertices on X.
4. Use 2 and 3, and the definition of δ to show $\beta(G) \geq |X| - \delta$.

Solution

By König's Theorem there is $\alpha'(G) = \beta(G)$, where $\beta(G)$ is a minimum vertex over. Hence we prove that

$$\beta(G) = |X| - \delta.$$

Let $S \subseteq X$ yielding

$$\delta = |S| - \Gamma(S).$$

Set a vertex cover $Z = \Gamma(S) \cup (X - S)$. Hence

$$\beta(G) \leq |Z| = |\Gamma(S)| + |X| - |S| = |X| - \delta.$$

To show the opposite inequality let Z be a minimum vertex cover and $S = X - Z$.

S is the vertices on X but out of Z. Hence their opposite vertices (neighbors in Y) must be in the cover Z. Hence

$$\Gamma(S) \subseteq Z \cap Y,$$

and

$$(1) \quad |\Gamma(S)| \leq |Z \cap Y|$$

To obtain the cover on Y from Z we need to remove from Z its vertices on X. Hence

$$(2) \quad |Z \cap Y| = |Z - (Z \cap X)| = |Z| - |Z \cap X|.$$

There is

$$(3) \quad |Z \cap X| = |X - (X - Z)| = |X| - |X - Z| = |X| - |S|.$$

Substitution of (2) and (3) into (1) yields

$$|\Gamma(S)| \leq |Z| - |X| + |S| = \beta(G) - |X| + |S|,$$

or

$$\beta(G) \geq |X| - (|S| - |\Gamma(S)|) \geq |X| - \max_{S \subseteq X}\{|S| - \Gamma(S)\} = |X| - \delta.$$