Problem 1 (60 Pts).

1. (20 Pts, proved in class) Prove that a digraph $G(V,E)$ is strongly connected iff $\forall X \subset V, X \neq \emptyset, \exists e(u,v) \in E, u \in X, v \in V \backslash X$.

Hint: proved in class.

2. (40 Pts) Let $G(V,E)$ be a digraph, and let $x, y \in V$. Suppose that all the arcs of E are colored red and black. Prove that <u>one and only one</u> of the following assertions holds:
   a) There is a path $P(x,y)$ in G of black and red arcs such that all the black arcs are oriented from x to y.
   b) $\exists X \subset V$ such that $\forall u \in X, v \in V \backslash X$ there is neither red $e(u,v) \in [X, V \backslash X]$ nor red $e(v,u) \in [X, V \backslash X]$, and there is no black arc $e(u,v) \in [X, V \backslash X]$.

Hints and guidance:

1. Obtain $G'\left(V', \vec{E}'\right)$ by contraction of all the red arcs of G.
2. Suppose first that $\nexists P(x,y)$ directed path in G' and apply part 1.
3. Then expand G' back to G and consider where the red arcs can (and cannot) be located.
4. For the <u>one and only one</u> of a) and b) assume the both exist and conclude a contradiction.

Proof of 1.2

Let $G'(V', \vec{E}')$ be a digraph obtained by the contraction the red arcs. G' thus has only black arcs.

We show first that $\nexists$ a) $\Rightarrow \exists$ b) . Suppose first $\nexists$ a) $\Rightarrow \nexists P(x, y)$ in G' where all arcs are oriented from x to y. By 1 above there is $X' \subset V'$, $X' \neq \emptyset$, such that $[X', V' \backslash X'] = \emptyset$ in the orientation from X' to $V' \backslash X'$r.

The vertices of X' and $V' \backslash X'$ either exist in V or have been obtained by contraction. Let us call the latter red vertices.

Let X'' be obtained from X' by expending the red vertices back into their origin, and expand similarly $V' \backslash X'$, yielding $V \backslash X''$.

Firstly, no black arc form X' to $V' \backslash X'$ $\Rightarrow$ no black arc form X'' to $V'' \backslash X''$.

Secondly, no red arc connecting X'' with $V \backslash X''$ in any direction since red arc yields red vertex, implying that red arcs are contained within $G[X'']$ and $G[V \backslash X'']$, but none in $[X'', V \backslash X'']$.

Trivially, if $\exists$ $P(x, y)$ in G' where all arcs are oriented from x to y, then a) is satisfied.

To show that one and only one of the above is possible assume that both a) and b) hold. Namely, there is a path $P(x, y)$ where all black arcs are oriented from x to y, and there is such X as in b).

Consider $e(u, v) \in P(x, y)$ s.t. $u \in X$, $v \in V \backslash X$. It follows from b) that $e(u, v)$ must be black. It follows also from b) that it cannot be oriented from X to $V \backslash X$. It cannot be oriented oppositely either because of a).

Problem 2 (60 Pts).

1. (40 Pts) Given $G(V, E)$ suppose $|V| = n$ and V has a partition $\{V_1, V_2, \dots, V_k\}$ such that $\forall\, 1 \leq i < j \leq k \quad \exists \quad x \in V_i \;,\; y \in V_j$ and $\nexists e(x, y)$ (no edge connects x and y). Show that

$$(1) \qquad \chi(G) \leq n - k + 1.$$

Hints and guidance:

1. Use induction on k to derive an upper bound of $\chi(G - V_k)$.
2. Extend this coloration to G by coloration of V_k and derive an upper bound of $\chi(G)$.
3. Get rid of the excessive color in 2 as follows.
4. Apply for V_k the property that $\forall\, 1 \leq i < j \leq k \;\exists\; x \in V_i \;,\; y \in V_j$ and $\nexists e(x, y)$ to derive a set $Y \subset V_k$, $Y = \{y_1, \dots, y_{k-1}\}$.
5. Show that there must be a color used for G which occurs only in Y.

2. (20 Pts) Show that

$$\chi(G) + \chi(\bar{G}) \leq n + 1.$$

Hints and guidance:

1. Let $\chi(\bar{G}) = k$ and consider a coloration $\{V_1, V_2, \dots, V_k\}$ of $\bar{G}$.
2. Prove that the conditions of problem 1 are satisfied by G.
3. Then use equation (1).

Proof of 2.1

By induction on k.

Suppose that

$$\chi(G - V_k) \leq (n - |V_k|) - (k - 1) + 1 = n - |V_k| - k + 2.$$

There is a coloration of $G - V_k$ by $n - |V_k| - k + 2$ colors.

It is possible to extend this coloration to G by using $|V_k|$ new colors, yielding a coloration of G by $n - k + 2$ colors.

By assumption $\forall\ 1 \leq i \leq k - 1\ \exists\ y_i \in V_i$ and a corresponding $x_i \in V_k$ such that $\nexists e(y_i, x_i)$.

Consider the $k - 1$ vertices $Y = \{y_1, \dots, y_{k-1}\}$. There must be color among the $n - k + 2$ colors used for G which occurs only in this set. This follows since $|V - Y| = n - k + 1$, whereas the coloration of G uses $n - k + 2$ colors.

Let $y_i \in Y$. Since $\nexists e(x_i, y_i)$ y_i can use the color of x_i which has already used in $G - V_k$ coloration.

The coloration of the rest of V_k requires no more the $|V_k| - 1$ new colors hence coloration of G by $n - k + 1$ colors.

Proof of 2.2

Let $\chi(\bar{G}) = k$ and consider a coloration $\{V_1, V_2, \dots, V_k\}$ of $\bar{G}$.

$\forall\ 1 \leq i < j \leq k\ \exists\ x \in V_i\ ,\ y \in V_j$ and $\exists e(x, y)$ (an edge must exist between V_i and V_j). Otherwise, two sets could be merged and $\chi(\bar{G}) < k$.

Hence the condition $\forall\ 1 \leq i < j \leq k\ \exists\ x \in V_i\ ,\ y \in V_j$ and $\nexists e(x, y)$ exists in G and the consequence of 2.1 holds.

$$\chi(G) \leq n - k + 1 = n - \chi(\bar{G}) + 1,$$

$$\chi(G) + \chi(\bar{G}) \leq n + 1.$$