**Problem 1 (60 Pts).**

1. (20 Pts, proved in class) Prove that a digraph is strongly connected iff , , , , .

Hint: proved in class.

1. (40 Pts) Let be a digraph, and let . Suppose that all the arcs of are colored red and black. Prove that one and only one of the following assertions holds:
2. There is a path in of black and red arcs such that all the black arcs are oriented from to .
3. such that there is neither red nor red , and there is no black arc .

Hints and guidance:

1. Obtain by contraction of all the red arcs of .
2. Suppose first that directed path in and apply part 1.
3. Then expand back to and consider where the red arcs can (and cannot) be located.
4. For the one and only one of a) and b) assume the both exist and conclude a contradiction.

**Proof of 1.2**

Let be a digraph obtained by the contraction the red arcs. thus has only black arcs.

We show first that . Suppose first ⇒ in where all arcs are oriented from to . By 1 above there is , , such that in the orientation from to r.

The vertices of and either exist in or have been obtained by contraction. Let us call the latter red vertices.

Let be obtained from by expending the red vertices back into their origin, and expand similarly , yielding .

Firstly, no black arc form to ⇒ no black arc form to .

Secondly, no red arc connecting with in any direction since red arc yields red vertex, implying that red arcs are contained within and , but none in .

Trivially, if in where all arcs are oriented from to , then a) is satisfied.

To show that one and only one of the above is possible assume that both a) and b) hold. Namely, there is a path where all black arcs are oriented from to , and there is such as in b).

Consider s.t. , . It follows from b) that must be black. It follows also from b) that it cannot be oriented from to . It cannot be oriented oppositely either because of a).

**Problem 2 (60 Pts).**

1. (40 Pts) Given suppose and has a partition such that , and (no edge connects and ). Show that
2. .

Hints and guidance:

1. Use induction on to derive an upper bound of .
2. Extend this coloration to by coloration of and derive an upper bound of .
3. Get rid of the excessive color in 2 as follows.
4. Apply for the property that , and to derive a set , .
5. Show that there must be a color used for which occurs only in .
6. (20 Pts) Show that

.

Hints and guidance:

1. Let and consider a coloration of .
2. Prove that the conditions of problem 1 are satisfied by .
3. Then use equation (1).

**Proof of 2.1**

By induction on .

Suppose that

.

There is a coloration of by colors.

It is possible to extend this coloration to by using new colors, yielding a coloration of by colors.

By assumption and a corresponding such that .

Consider the vertices . There must be color among the colors used for which occurs only in this set. This follows since , whereas the coloration of uses colors.

Let . Since can use the color of which has already used in coloration.

The coloration of the rest of requires no more the new colors hence coloration of by colors.

**Proof of 2.2**

Let and consider a coloration of .

, and (an edge must exist between and ). Otherwise, two sets could be merged and .

Hence the condition , and exists in and the consequence of 2.1 holds.

,

.