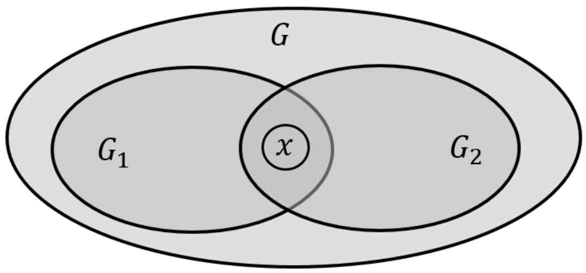
Question 1 (Connectivity 60 Pts).

Prove that a connected bipartite k–regular graph is 2-connected.

Hints:

1. Prove by contradiction, consider the graph structure below.
2. Then consider the degree of x in one of the subgraph and recall that any subgraph is still bipartite.



Question 2 (Coloring 60 Pts).

Let G be $(k+1)$ critical (every $G' \subset G$ is k–colorable). Prove that G is at least k – edge-connected (deletion of less than k edges leaves G connected).

Hint:

1. Prove by contradiction, separate G into two pieces by edge deletion.
2. Color each piece and then match the colors of the two pieces such that G is colored properly.

Proof 1: Assume in contrary that G is not 2-connected, hence it is 1-connected.

Therefore, G can be decomposed into two subgraphs G_1 and G_2 sharing a single vertex $x \in V(G)$, namely, $G = G_1 \cup G_2$, $V(G_1) \cap V(G_2) = \{x\}$, $|V(G_1)| \geq 2$ and $|V(G_2)| \geq 2$.

Since $d_G(x) = k$ and x is a disconnecting vertex, it must have neighbors in both G_1 and G_2, hence $1 \leq d_{G_1}(x) \leq k - 1$.

Since G is 1-connected, there is no edge connecting $V(G_1) - \{x\}$ to $V(G_2) - \{x\}$, hence $d_{G_1}(y) = k \ \forall y \in V(G_1) - \{x\}$.

Since G is bipartite, let $V(G_1) = R \cup S$, where $R = \{x, u_1, u_2, \dots, u_r\}$ and $S = \{v_1, v_2, \dots, v_s\}$ are in the two color classes of G, namely all $E(G_1)$ connect only vertices of R to S. Hence $|E(G_1)| = \sum_{y \in R} d_{G_1}(y) = \sum_{y \in S} d_{G_1}(y)$.

$d_{G_1}(x) + kr = \sum_{y \in R} d_{G_1}(y) = \sum_{y \in S} d_{G_1}(y) = ks$.

It follows that the right hand side is divisible by k whereas the left hand sides does not, hence a contradiction.

Proof 2: Assume in contrary that the deletion of $m < k$ edges e_1,…, e_m separates G into two components G_1 and G_2, and let m be the smallest such number.

It follows that $\forall e_i$, $1 \leq i \leq m$, the end vertices of e_i belong to G_1 and G_2.

Since G is $k + 1$ critical, G_1 and G_2 are k–colorable. So let T_1,…, T_k and S_1,…, S_k be the color classes of G_1 and G_2 , respectively, i.e., $V(G_1) = \bigcup_{1 \leq i \leq k} T_i$ and $V(G_2) = \bigcup_{1 \leq i \leq k} S_i$.

We would like to match some T_i with some S_i, such that for none of e_1,…, e_m the same color appears on its two end vertices, thus ensuring k –proper coloring in contradiction of its $k + 1$ criticality.

Since $m < k$, there is an S_i among S_1,…, S_k not connected to T_1 by any edge of e_1,…, e_m.

We can certainly select this S_i such that either T_1 or S_i should be incident with some edge of e_1,…, e_m. So let us call this color class S'_1.

We are left with a matching problem of $k-1$ T color classes with $k-1$ S color classes and $m-1<k-1$. The above process can be repeated until all e_1,…, e_m are consumed.