**Question 1** (Connectivity 60 Pts).

Prove that a connected bipartite $k$–regular graph is $2$-connected.

Hints:

1. Prove by contradiction, consider the graph structure below.
2. Then consider the degree of $x$ in one of the subgraph and recall that any subgraph is still bipartite.



**Question 2** (Coloring 60 Pts).

Let $G$ be $\left(k+1\right)$ critical (every $G^{'}⊂G$ is $k$–colorable). Prove that $G$ is at least $k$ –edge-connected (deletion of less than $k$ edges leaves $G$ connected).

Hint:

1. Prove by contradiction, separate $G$ into two pieces by edge deletion.
2. Color each piece and then match the colors of the two pieces such that $G$ is colored properly.

**Proof 1**: Assume in contrary that $G$ is not $2$-connected, hence it is $1$-connected.

Therefore, $G$ can be decomposed into two subgraphs $G\_{1}$ and $G\_{2}$ sharing a single vertex $x\in V\left(G\right)$, namely, $G=G\_{1}∪G\_{2}$ , $V\left(G\_{1}\right)∩V\left(G\_{2}\right)=\left\{x\right\}$ , $\left|V\left(G\_{1}\right)\right|\geq 2$ and $\left|V\left(G\_{2}\right)\right|\geq 2$.

Since $d\_{G}\left(x\right)=k$ and $x$ is a disconnecting vertex, it must have neighbors in both $G\_{1}$ and $G\_{2}$, hence $1\leq d\_{G\_{1}}\left(x\right)\leq k-1$.

Since $G$ is $1$-connected, there is no edge connecting $V\left(G\_{1}\right)-\left\{x\right\}$ to $V\left(G\_{2}\right)-\left\{x\right\}$ , hence $d\_{G\_{1}}\left(y\right)=k$ $∀y\in V\left(G\_{1}\right)-\left\{x\right\}$.

Since $G$ is bipartite, let $V\left(G\_{1}\right)=R∪S$, where $R=\left\{x,u\_{1},u\_{2}, …, u\_{r}\right\}$ and $S=\left\{v\_{1},v\_{2}, …, v\_{s}\right\}$ are in the two color classes of $G$, namely all $E\left(G\_{1}\right)$ connect only vertices of $R$ to $S$. Hence $\left|E\left(G\_{1}\right)\right|=\sum\_{y\in R}^{}d\_{G\_{1}}\left(y\right)=\sum\_{y\in S}^{}d\_{G\_{1}}\left(y\right)$.

$d\_{G\_{1}}\left(x\right)+kr=\sum\_{y\in R}^{}d\_{G\_{1}}\left(y\right)=\sum\_{y\in S}^{}d\_{G\_{1}}\left(y\right)=ks$.

It follows that the right hand side is divisible by $k$ whereas the left hand sides does not, hence a contradiction.

**Proof 2**: Assume in contrary that the deletion of $m<k$ edges $e\_{1}$,…,$ e\_{m}$ separates $G$ into two components $G\_{1}$ and $G\_{2}$, and let $m$ be the smallest such number.

It follows that $∀e\_{i}, 1\leq i\leq m$ , the end vertices of $e\_{i}$ belong to $G\_{1}$ and $G\_{2}$.

Since $G$ is $k+1$ critical, $G\_{1}$ and $G\_{2}$ are $k–$colorable. So let $T\_{1}$,…,$ T\_{k}$ and $S\_{1}$,…,$ S\_{k}$ be the color classes of $G\_{1}$ and $G\_{2}$ , respectively, i.e., $V\left(G\_{1}\right)=\bigcup\_{1\leq i\leq k}^{}T\_{i}$ and $V\left(G\_{2}\right)=\bigcup\_{1\leq i\leq k}^{}S\_{i}$.

We would like to match some $T\_{i}$ with some $S\_{i}$, such that for none of $e\_{1}$,…,$ e\_{m}$ the same color appears on its two end vertices, thus ensuring $k-$proper coloring in contradiction of its $k+1$ criticality.

Since $m<k$, there is an $S\_{i}$ among $S\_{1}$,…,$ S\_{k}$ not connected to $T\_{1}$ by any edge of $e\_{1}$,…,$ e\_{m}$.

We can certainly select this $S\_{i}$ such that either $T\_{1}$ or $S\_{i}$ should be incident with some edge of $e\_{1}$,…,$ e\_{m}$. So let us call this color class $S'\_{1}$.

We are left with a matching problem of $k-1$ $T$ color classes with $k-1$ $S$ color classes and $m-1<k-1$. The above process can be repeated until all $e\_{1}$,…,$ e\_{m}$ are consumed.