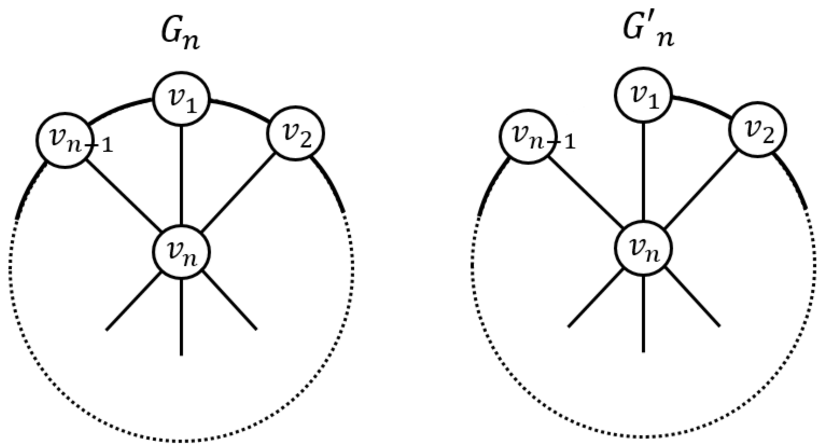
Question 1 (Chromatic polynomial 40 Pts).

Consider the following n-vertex wheel graphs G_n and G'_n, where $d(v_n) = n - 1$ and G'_n is obtained by removing an edge of G_n.



Prove the following:

1. (5Pts) $P_{G'_n}(k) = k(k-1)(k-2)^{n-2}$
2. (10Pts) $P_{G_n}(k) = P_{G'_n}(k) - P_{G_{n-1}}(k)$
3. (15Pts) $P_{G_n}(k) = k(k-2)[(k-2)^{n-2} + (-1)^{n-1}]$ for $n \geq 4$ (use induction).

Let $P_G(k) = k^4 - 4k^3 + 5k^2 - 2k$.

4. (5Pts) How many vertices and edges G has?
5. (5Pts) Is G bipartite?

Question 2 (Graph connectivity 40 Pts).

1. (25 Pts.) Let G be connected such that its longest paths contain n vertices. Let P_1 and P_2 be such longest paths in G and $P_1 \neq P_2$. Prove that P_1 and P_2 must have a common vertex.
2. (15 Pts.) Let G be connected and e an edge of G . Suppose that every spanning tree of G contains e. Show that $G \setminus e$ (e deleted) is disconnected.

Question 3 (Cliques and independent sets 40 Pts).

Let $G(V, E)$ be such that $|V| = \binom{k+l}{k}$. Show that either $K_{k+1} \subset G$ ($k + 1$ vertex clique) or $K_{l+1} \subset \bar{G}$ ($l + 1$ vertex independent set).

Proof 3: By induction on $k+l$.

For $k=1$ or $l=1$ the assertion is obvious.

Suppose that $k,l>1$. Let $v \in V$. There is

$$d_G(v)+d_{\bar{G}}(v)=|V|-1=\binom{k+l}{k}-1.$$

There is

$$\binom{k+l}{k}=\frac{(k+l)!}{k!\,l!}=\frac{(k+l-1)!}{k!\,l!}\times(k+l)=\frac{(k+l-1)!}{(k-1)!\,l!}+\frac{(k+l-1)!}{k!\,(l-1)!}=$$
$$\binom{k+l-1}{k-1}+\binom{k+l-1}{k}.$$

Substitution yields

$$d_G(v)+d_{\bar{G}}(v)=\binom{k+l-1}{k-1}+\binom{k+l-1}{k}-1$$

Hence either

(a) $d_G(v)\geq\binom{k+l-1}{k-1}$ or (b) $d_{\bar{G}}(v)\geq\binom{k+l-1}{k}$.

If case (a) $d_G(v)\geq\binom{k+l-1}{k-1}$ holds, let $G_1\subset G$ be induced the neighbors of v. By induction hypothesis either $K_k\subset G_1$ or $K_{l+1}\subset\overline{G_1}$.

For $K_{l+1}\subset\overline{G_1}\subset\bar{G}$ we are done.

For $K_k\subset G_1$, the vertex set $V(K_k)\cup\{v\}\subset V$ spans K_{k+1} in G hence $K_{k+1}\subset G$.

Same conclusions follow if case (b) $d_{\bar{G}}(v)\geq\binom{k+l-1}{k}=\binom{k+l-1}{l-1}$ holds. ■