

February 2021

## Final Exam in Graph Theory

Answer the two questions.

Total number of points is 140, maximum grade is 100.

Exam period is two hours (120 minutes).

Write your answers clearly. Submit by email as PDF to [wimers@biu.ac.il](mailto:wimers@biu.ac.il)

Good luck!

### Question 1 (70 Pts)

Let  $G(V, E)$  have  $n$  vertices and  $m$  edges. Prove that its chromatic polynomial  $P_G(k)$  satisfies:

- Its degree is  $n$ .
- The coefficient of  $k^n$  is 1.
- The coefficient of  $k^{n-1}$  is  $m$ .
- The sign of the coefficients is alternating.
- The free coefficient is zero.

**Hint:** You may use induction on edges and the underlying idea in the Theorem of how to derive the chromatic polynomial of a graph.

### Solution

Use induction on the number  $r$  of edges.

For  $r = 0$  there is  $P_G(k) = k^n$  and all the above statements hold.

Assume that the statements hold for  $r < m$  and let  $E(G) = m$ .

Let  $e \in E(G)$ ,  $e = st$ . In any proper coloring  $f \rightarrow E(G)$  there is  $f(s) \neq f(t)$ .

Consider  $G_1 = G - e$ . There is  $E(G_1) = m - 1$  and the induction hypothesis applies, hence

$$P_{G_1}(k) = k^n - (m - 1)k^{n-1} + b_1k^{n-2} + \cdots \pm z_1k.$$

Proper colorings of  $G_1$  include also all the cases where  $f(s) = f(t)$ , so we have to subtract them.

The total number of  $G_1$  coloring where  $f(s) = f(t)$  is the chromatic polynomial obtained by contraction of  $e = st$  in  $G$ , yielding a graph  $G_2$ ,  $E(G_2) = m - 1$  and  $V(G_2) = n - 1$ . By induction

$$P_{G_2}(k) = k^{n-1} - (m-1)k^{n-2} + \dots \pm z_2 k.$$

All in all,

$$P_G(k) = P_{G_1}(k) - P_{G_2}(k) = k^n - mk^{n-1} + \dots \pm (z_1 + z_2)k.$$

The alternating signs are preserved since terms of same power in  $P_{G_1}(k)$  and  $P_{G_2}(k)$  have opposite signs. ■

## Question 2 (70 Pts)

Menger's Theorem (studied in the course) holds also for two vertex subsets  $X \subset V(G)$  and  $Y \subset V(G)$ ,  $|X| \geq 1$  and  $|Y| \geq 1$ , namely,  $p(X, Y) = c(X, Y)$ .

**You don't need to prove it.** Rather use it to prove the following:

Let  $G(V, E)$  be  $k$ -connected,  $k \geq 2$ . Prove that for any  $S \subset V(G)$ ,  $|S| = k$ , there exists a cycle which contains these  $k$  vertices.

### Guidance for solution:

- Use induction on  $k$ .
- Consider a vertex subset  $S \subset V(G)$ ,  $|S| = k$ . Then remove from  $G$  a vertex  $v \in S$ . What is the vertex connectivity of  $G - v$ ?
- Apply the induction for  $G - v$ .
- Let  $X = S - v$  and  $Y = v$  in above Menger's Theorem to show that there is a cycle in  $G$  containing  $S$ .

### Solution

By Menger Theorem for  $k = 2$  there are two internally disjoint paths connecting any two vertices, hence a cycle exists.

Let  $S \subset V(G)$ ,  $|S| = k$ . Let  $v \in S$ . Since  $G$  is  $k$ -connected  $G - v$  is  $(k - 1)$ -connected.

Hence, by induction hypothesis there is a cycle  $C$  in  $G - v$  containing the vertices of  $S - v$ . W.L.O.G denote them  $\{v_1, v_2, \dots, v_{k-1}\}$  by their order along the cycle.

Let  $X = C - v$  and  $Y = v$  (it works also for  $X = S - v$ ). By Menger Theorem there are  $k$  internally disjoint paths connecting  $v$  to  $C - v$ .

The cycle  $C$  has  $k - 1$  sections  $[v_i, \dots, v_{i+1}]$ , hence at least two paths  $P_1$  and  $P_2$  of the  $k$  internally disjoint paths connecting  $v$  to  $C - v$  must end at the same section as illustrated below.

The cycle  $v \rightarrow P_1 \rightarrow C \rightarrow P_2 \rightarrow v$  contains all the vertices of  $S$ . ■

