February 2021

**Final Exam in Graph Theory**

Answer the two questions.

Total number of points is 140, maximum grade is 100.

Exam period is two hours (120 minutes).

Write your answers clearly. Submit by email as PDF to [wimers@biu.ac.il](mailto:wimers@biu.ac.il)

Good luck!

**Question 1 (70 Pts)**

Let have vertices and edges. Prove that its chromatic polynomial satisfies:

* Its degree is.
* The coefficient of is 1.
* The coefficient of is .
* The sign of the coefficients is alternating.
* The free coefficient is zero.

**Hint**: You may use induction on edges and the underlying idea in the Theorem of how to derive the chromatic polynomial of a graph.

**Solution**

Use induction on the number of edges.

For there is and all the above statements hold.

Assume that the statements hold for and let .

Let . In any proper coloring there is .

Consider . There is and the induction hypothesis applies, hence

.

Proper colorings of include also all the cases where , so we have to subtract them.

The total number of coloring where is the chromatic polynomial obtained by contraction of in , yielding a graph , and . By induction

.

All in all,

.

The alternating signs are preserved since terms of same power in and have opposite signs.

**Question 2 (70 Pts)**

Menger’s Theorem (studied in the course) holds also for two vertex subsets and , and , namely, . **You don’t need to prove it**. Rather use it to prove the following:

Let be -connected, . Prove that for any . there exists a cycle which contains these vertices.

**Guidance for solution**:

* Use induction on .
* Consider a vertex subset , . Then remove from a vertex . What is the vertex connectivity of ?
* Apply the induction for .
* Let and in above Menger’s Theorem to show that there is a cycle in containing .

**Solution**

By Manger Theorem for there are two internally disjoint paths connecting any two vertices, hence a cycle exists.

Let , . Let . Since is -connected is -connected.

Hence, by induction hypothesis there is a cycle in containing the vertices of . W.L.O.G denote them by their order along the cycle.

Let and (it works also for ). By Menger Theorem there are internally disjoint paths connecting to .

The cycle has sections , hence at least two paths and of the internally disjoint paths connecting to must end at the same section as illustrated below.

The cycle contains all the vertices of .

