

Linear Dynamical Systems

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Linear dynamical model:

$$\begin{aligned} X_t &= AX_{t-1} + v_t & v_t &\sim N(0, Q) & X_1 &\sim N(\mu_0, \Sigma_0) \\ Y_t &= CX_t + w_t & w_t &\sim N(0, R) \end{aligned}$$

Notation :

$$\begin{aligned} x_{t|s} &= E(x_t | y_1, \dots, y_s) \\ P_{t|s} &= V(x_t | y_1, \dots, y_s) = E((x_t - x_{t|s})(x_t - x_{t|s})^\top | y_1, \dots, y_s) \\ P_{t,t-1|s} &= E((x_t - x_{t|s})(x_{t-1} - x_{t-1|s})^\top | y_1, \dots, y_s) \end{aligned}$$

Kalman Filter:

$$\begin{aligned} x_{1|0} &= \mu_0 & t &= 1 \\ P_{1|0} &= \Sigma_0 \\ P_{t|t-1} &= AP_{t-1|t-1}A^\top + Q & t &= 2, \dots, n \\ x_{t|t-1} &= Ax_{t-1|t-1} \\ K_t &= P_{t|t-1}C^\top (CP_{t|t-1}C^\top + R)^{-1} \\ P_{t|t} &= (I - K_tC)P_{t|t-1}(I - K_tC)^\top + K_tRK_t^\top \\ x_{t|t} &= x_{t|t-1} + K_t(y_t - Cx_{t|t-1}) \end{aligned}$$

Kalman Smoother:

$$\begin{aligned} J_t &= P_{t|t}A^\top P_{t+1|t}^{-1} & t &= n-1, \dots, 1 \\ P_{t|n} &= P_{t|t} + J_t(P_{t+1|n} - P_{t+1|t})J_t^\top \\ x_{t|n} &= x_{t|t} + J_t(x_{t+1|n} - x_{t+1|t}) \end{aligned}$$

Log Likelihood of y_1, \dots, y_n :

$$\begin{aligned} \log f(y_1, \dots, y_n) &= \sum_t \log f(y_t | y_1, \dots, y_{t-1}) \\ y_t | y_1, \dots, y_{t-1} &\sim N(Cx_{t|t-1}, CP_{t|t-1}C^\top + R) \end{aligned}$$

Lag One Covariance Smoother:

$$\begin{aligned}
 P_{t-1,t|t} &= (I - K_t C) A P_{t-1|t-1} & t = 2, \dots, n \\
 P_{t-1,t|n} &= (I + (P_{t|n} - P_{t|t}) P_{t|t}^{-1}) P_{t-1,t|t} & t = n, \dots, 2
 \end{aligned}$$

EM iteration:

- Expectation step:

Perform Kalman smoother and lag one covariance smoother and compute the following:

$$\begin{aligned}
 \langle y_t y_t^\top \rangle &= \frac{1}{n} \sum_t y_t y_t^\top \\
 \langle y_t x_t^\top \rangle &= \frac{1}{n} \sum_t y_t x_{t|n}^\top \\
 \langle x_t x_t^\top \rangle_{n_1, n_2} &= \frac{1}{n_2 - n_1 + 1} \sum_{t=n_1}^{n_2} P_{t|n} + x_{t|n} x_{t|n}^\top \\
 \langle x_{t-1} x_t^\top \rangle &= \frac{1}{n-1} \sum_t P_{t-1,t|n} + x_{t-1|n} x_{t|n}^\top
 \end{aligned}$$

- Maximization step:

$$\begin{aligned}
 \hat{C} &= \langle y_t x_t^\top \rangle \langle x_t x_t^\top \rangle_{1,n}^{-1} \\
 \hat{R} &= \langle y_t y_t^\top \rangle - \hat{C} \langle x_t y_t^\top \rangle \\
 \hat{A} &= \langle x_t x_{t-1}^\top \rangle \langle x_t x_t^\top \rangle_{1,n-1}^{-1} \\
 \hat{Q} &= \langle x_t x_t^\top \rangle_{2,n} - \hat{A} \langle x_{t-1} x_t^\top \rangle \\
 \hat{\mu}_0 &= x_{1|n} \\
 \hat{\Sigma}_0 &= P_{1|n}
 \end{aligned}$$