Hidden Markov Model

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Generative model:

$$\begin{array}{rcll} X_1 & & p(X_1 = j) = p(j) & \\ X_t \leftarrow X_{t-1} & & p(X_t = j | X_{t-1} = i) = p(j|i) & i, j = 1, ..., k \\ Y_t | X_t = j & \sim & N(\mu_j, \Sigma_j) & \end{array}$$

$$f(y_1, ..., y_n) = \sum_{x_1, ..., x_n} p(x_1) p(y_1|x_1) \prod_{t=2}^{n} p(x_t|x_{t-1}) p(y_t|x_t)$$

Forward-Backward Algorithm:

$$\alpha_t(j) = p(y_1, ..., y_t, x_t = j) \qquad \beta_t(j) = p(y_{t+1}, ..., y_n | x_t = j)$$

$$\begin{array}{rcll} \alpha_1(j) & = & p(j) \cdot p(y_1|x_1 = j) & \\ \alpha_t(j) & = & \sum_{i=1}^{k} \alpha_{t-1}(i) \cdot p(j|i) \cdot p(y_t|x_t = j) & t = 2, ..., n \\ \beta_n(j) & = & 1 & \\ \beta_t(j) & = & \sum_{i=1}^{k} p(i|j) \cdot p(y_{t+1}|x_{t+1} = i) \cdot \beta_{t+1}(i) & t = n-1, .., 1 \end{array}$$

Viterbi Algorithm: $\hat{x}_1, ..., \hat{x}_n = \arg\max_x p(x_1, ..., x_n | y_1, ..., y_n)$

$$\begin{array}{rcll} v_1(j) & = & \log p(j) + \log p(y_1|x_1 = j) & \\ v_t(j) & = & \max_{i=1}^{k} \{v_{t-1}(i) + \log p(j|i) + \log p(y_t|x_t = j)\} & t = 2, ..., n \\ \delta_t(j) & = & \arg\max_{i=1}^{k} \{v_{t-1}(i) + \log p(j|i)\} & \\ \hat{x}_n & = & \arg\max_{j=1}^{k} v_n(j) & \\ \hat{x}_t & = & \delta_{t+1}(x_{t+1}) & t = n-1, ..., 1 \end{array}$$

EM (Baum-Welch) iteration:

- Expectation step :

$$
\begin{aligned}
w_{t,j} &= p(x_t = j|y) = \frac{\alpha_t(j)\beta_t(j)}{f(y)} \\
w_{t,i,j} &= p(x_t = i, x_{t+1} = j|y) = \frac{\alpha_t(i)p(j|i)p(y_{t+1}|x_{t+1} = j)\beta_{t+1}(j)}{f(y)}
\end{aligned}
$$

- Maximization step :

$$
\begin{aligned}
\hat{p}(j|i) &\leftarrow \frac{\sum_{t=1}^{n-1} w_{tij}}{\sum_{t=1}^{n-1} w_{ti}} \\
\hat{\mu}_j &\leftarrow \frac{\sum_{t=1}^{n} w_{tj} y_t}{\sum_{t=1}^{n} w_{tj}} \\
\hat{\Sigma}_j &\leftarrow \frac{\sum_{t=1}^{n} w_{tj}(y_t - \hat{\mu}_j)(y_t - \hat{\mu}_j)^T}{\sum_{t=1}^{n} w_{tj}}
\end{aligned}
$$