Non-Linear Dynamical Systems

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Non-linear dynamical systems with Gaussian additive noise:

$$\begin{aligned} X_t &= f(X_{t-1}) + v_t \qquad & v_t \sim N(0,Q) \qquad & X_1 \sim N(\mu_0, \Sigma_0) \\ Y_t &= h(X_t) + w_t \qquad & w_t \sim N(0,R) & \end{aligned}$$

linear approximation:

$$\begin{aligned} f(x_{t-1}) &= f(x_{t-1|t-1}) + \frac{\partial f}{\partial x}(x_{t-1|t-1})(x_{t-1} - x_{t-1|t-1}) \\ h(x_t) &= h(x_{t|t-1}) + \frac{\partial h}{\partial x}(x_{t|t-1})(x_t - x_{t|t-1}) \end{aligned}$$

Extended Kalman Filter (EKF):

$$\begin{aligned} A_t &= \frac{\partial f}{\partial x}(x_{t-1|t-1}) \\ P_{t|t-1} &= A_t P_{t-1|t-1} A_t^T + Q \\ x_{t|t-1} &= f(x_{t-1|t-1}) \\ C_t &= \frac{\partial h}{\partial x}(x_{t|t-1}) \\ K_t &= P_{t|t-1} C_t^T (C_t P_{t|t-1} C_t^T + R)^{-1} \\ P_{t|t} &= (I - K_t C_t) P_{t|t-1} (I - K_t C_t)^T + K_t R K_t^T \\ x_{t|t} &= x_{t|t-1} + K_t (y_t - h(x_{t|t-1})) \end{aligned}$$

General non-linear dynamical system:

$$\begin{aligned} X_t &= f(X_{t-1}, v_t) \qquad & p(X_1), p(X_t|X_{t-1}) \\ Y_t &= h(X_t, w_t) \qquad & p(Y_t|X_t) \end{aligned}$$

Particle Filter:

sample $x_t(i) \sim q(x_t|x_{t-1}(i), y_t)$ $\qquad$ $i = 1, ..., \text{n-particles}$

$$w(i) = \alpha \, \frac{p(x_t(i) \,|\, x_{t-1}(i)) \, p(y_t|x_t(i))}{q(x_t(i) \,|\, x_{t-1}(i), y_t)}$$

$$x_{t|t} = \sum_i w(i) x_t(i)$$

$$P_{t|t} = \sum_i w(i)(x_t(i) - x_{t|t})(x_t(i) - x_{t|t})^T$$

re-sampling: let D be the discrete probability $\{x_t(i), w(i)\}$

resample $x_t(i)$ $\quad$ from D $\qquad$ $i = 1, ..., \text{n-particles}$

Non-linear dynamical systems with Gaussian additive noise:

$$
\begin{aligned}
X_t &= f(X_{t-1}) + v_t & v_t &\sim N(0, Q) & X_1 &\sim N(\mu_0, \Sigma_0) \\
Y_t &= h(X_t) + w_t & w_t &\sim N(0, R)
\end{aligned}
$$

Unscented Kalman Filter (UKF):

$$
\begin{aligned}
x_{1|0}(i) &= f(\mu_0 + (\sqrt{2k \cdot \Sigma_0})_i) & i &= 1, ..., k \\
x_{1|0}(k+i) &= f(\mu_0 - (\sqrt{2k \cdot \Sigma_0})_i) & i &= 1, ..., k \\
x_{1|0}(2k+i) &= f(\mu_0) & i &= 1, ..., k \\
x_{1|0}(3k+i) &= f(\mu_0) & i &= 1, ..., k
\end{aligned}
$$

$$
\begin{aligned}
x_{t|t-1}(i) &= f(x_{t-1|t-1} + (\sqrt{2k \cdot P_{t-1|t-1}})_i) & i &= 1, ..., k \\
x_{t|t-1}(k+i) &= f(x_{t-1|t-1} - (\sqrt{2k \cdot P_{t-1|t-1}})_i) & i &= 1, ..., k \\
x_{t|t-1}(2k+i) &= f(x_{t-1|t-1}) + (\sqrt{2k \cdot Q})_i & i &= 1, ..., k \\
x_{t|t-1}(3k+i) &= f(x_{t-1|t-1}) - (\sqrt{2k \cdot Q})_i & i &= 1, ..., k \\
x_{t|t-1} &= \frac{1}{4k} \sum_i x_{t|t-1}(i) \\
P_{t|t-1} &= \frac{1}{4k} \sum_i (x_{t|t-1}(i) - x_{t|t-1})(x_{t|t-1}(i) - x_{t|t-1})^T \\
y_{t|t-1}(i) &= h(x_{t|t-1}(i)) & i &= 1, ..., 4k \\
y_{t|t-1} &= \frac{1}{4k} \sum_i y_{t|t-1}(i) \\
P_{y_t, y_t} &= \frac{1}{4k} \sum_i (y_{t|t-1}(i) - y_{t|t-1})(y_{t|t-1}(i) - y_{t|t-1})^T + R \\
P_{x_t, y_t} &= \frac{1}{4k} \sum_i (x_{t|t-1}(i) - x_{t|t-1})(y_{t|t-1}(i) - y_{t|t-1})^T \\
K_t &= P_{x_t, y_t} \cdot P_{y_t, y_t}^{-1} \\
x_{t|t} &= x_{t|t-1} + K_t(y_t - y_{t|t-1}) \\
P_{t|t} &= P_{t|t-1} - K_t P_{y_t, y_t} K_t^T
\end{aligned}
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