Gaussian Mixture Model

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Generative model:

$$\begin{array}{ccll} X & & p(X=j)=\alpha_j & \qquad j=1,...,k \\ Y|X=j & \sim & N(\mu_j,\Sigma_j) & \end{array}$$

The distribution of a random variable $Y \in R^d$ is a mixture of k Gaussians if :

$$f(Y=y|\theta) = \sum_{j=1}^{k} \alpha_j \frac{1}{\sqrt{(2\pi)^d |\Sigma_j|}} \exp\{-\frac{1}{2}(y-\mu_j)^T \Sigma_j^{-1} (y-\mu_j)\}$$

such that $\theta = \{\alpha_j, \mu_j, \Sigma_j\}_{j=1}^k$ consists of :

- $\alpha_j > 0 \quad j=1,...k \qquad \sum_{j=1}^k \alpha_j = 1$
- $\mu_j \in R^d$ and Σ_j is a $d \times d$ positive definite matrix $\quad j=1,...,k$

Likelihood of $y_1, \ldots, y_n$:

$$f(y_1,...,y_n|\theta) = \prod_{t=1}^{n} \sum_{j=1}^{k} \alpha_j f(y_t|\mu_j,\Sigma_j)$$

EM iteration:

- Expectation step :

$$w_{tj} = p(x_t=j|y_t) = \frac{\alpha_j f(y_t|\mu_j,\Sigma_j)}{\sum_{i=1}^k \alpha_i f(y_t|\mu_i,\Sigma_i)}$$

- Maximization step :

$$\begin{array}{lcl} \hat{\alpha}_j & \leftarrow & \frac{1}{n}\sum_{t=1}^{n} w_{tj} \\ \hat{\mu}_j & \leftarrow & \frac{\sum_{t=1}^n w_{tj} y_t}{\sum_{t=1}^n w_{tj}} \\ \hat{\Sigma}_j & \leftarrow & \frac{\sum_{t=1}^n w_{tj}(y_t-\hat{\mu}_j)(y_t-\hat{\mu}_j)^T}{\sum_{t=1}^n w_{tj}} \end{array}$$