Sensitivity Analysis of MVDR and MPDR Beamformers

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Abstract—A sensitivity analysis of two distortionless beamformers is presented in this paper. Specifically, two well-known variants, namely the minimum power distortionless response (MPDR) and minimum variance distortionless response (MVDR) beamformers, are considered. In our scenario, which is typical to many modern communications systems, waves emitted by multiple point sources are received by an antenna array. An analytical expression for the signal to interference and noise ratio (SINR) improvement obtained by both beamformers under steering errors is derived. These expression are experimentally evaluated and compared with the robust Capon beamformer (RCB), a robust variant of the MPDR beamformer. We show that the MVDR beamformer, which uses the noise correlation matrix in its minimization criterion, is more robust to steering errors than its counterparts, that use the received signal correlation matrix. Furthermore, even if the noise correlation matrix is erroneously estimated due to steering errors in the interference direction, the MVDR advantage is still maintained for reasonable range of steering errors. These conclusions conform with Cox [1] findings. Only line of sight propagation regime is considered in the current contribution. Ongoing research extends this work to fading channels.

(Topic: Signal Processing)

I. INTRODUCTION

Beamformers are widely used in a variety of signal processing applications, e.g. speech enhancement, radar, and wireless communications. In particular, the problem of estimating signals of multiple transmitters using an antenna array is one of the fundamental problem in array processing. In this context, the Capon beamformer [2] gained popularity due to its high interference rejection capabilities. The Capon beamformer, also referred to by Van Trees as MPDR beamformer [3], is however sensitive to steering errors. Several robust techniques for alleviating the sensitivity problem were therefore proposed in the literature. Ward et al. [4] provides a comparison of several robust MPDR beamformers, based on diagonal loading, that compensates for model errors. Wax et al. [5] presents an analysis of the signal-to-interference-plus-noise ratio (SINR) for the MPDR beamformer, in terms of the signal-to-noise ratio (SNR), interference-to-noise ratio (INR), signal-to-interference ratio (SIR), angular separation between desired signal and interferences, and correlation between these signals. Reddy et al. [6] study the signal cancellation and interference rejection effects of the MPDR beamformer in the presence of correlated interfering sources.

Of special interest to our study, is the analysis presented by Cox [1]. In his seminal work Cox compares the sensitivity of two robust distortionless beamformers to steering errors. The two structures differ in the correlation matrix used. While the MVDR beamformer is using the noise correlation matrix, the MPDR beamformer is using the received signals correlation matrix (desired signal plus noise). The robustness of the two beamformers to steering errors in the desired signal direction is analyzed. It is argued that the MPDR beamformer, that incorporates the desired signal into the correlation matrix, tends to suppress the signal impinging the array from the desired source when it is erroneously steered. The application of the more robust MVDR beamformer necessitates the availability of an estimate of the noise correlation matrix.

In our work we consider scenarios in which multiple signals emanating from point sources are received by the array. The interfering sources together with noncoherent sources constitute the noise correlation matrix. We show that under uncertainties in the interference signal steering vectors the MVDR beamformer maintains higher robustness as compared with the MPDR beamformer.

The structure of this paper follows. In Section II the problem is formulated and both the MVDR and MPDR beamformers are introduced. An analytical expression for the SINR improvement of both structures is derived in Section III. These cumbersome expression are experimentally evaluated in Section IV and compared with the RCB algorithm, which is a robust variant of the MPDR beamformer. The experimental results are discussed and possible explanations for the beamformers behavior are given in the same section. The paper is concluded in Section IV-C.

II. PROBLEM FORMULATION

Consider L elements antenna array and denote the complex envelope of the received narrow-band signals as \( y[t] \in \mathbb{C}^{L \times 1} \). The received signals are related to the transmitted signal by

\[
y(t) = d(t)s(t) + G(t)s_1(t) + n(t)
\]

where, the desired signal is denoted \( s \), its power is \( \sigma_d^2 \), and \( d \) is the corresponding steering vector. \( s = [s_1, \ldots, s_K] \) is a vector of \( K \) uncorrelated interference signals, \( A = \text{diag}(\sigma_1^2, \ldots, \sigma_K^2) \) is its corresponding correlation matrix, and \( G = [g_1, \ldots, g_K] \) is a matrix consisting of the corresponding steering vectors. \( n \) are spatially-white sensor noise signals with correlation matrix \( \sigma_n^2 \). For the sake of brevity, we have omitted the time index. We will adopt this convention whenever no confusion can arise.

The correlation matrix of the received signal \( y \) is given by

\[
R_y = R_s + R_n
\]

where \( R_s = \sigma_d^2 dd^H \) is the desired signal component, and \( R_n = GA^H + \sigma_n^2 I \) is the interference signals component. Defining \( B = \frac{G}{\sigma_d^2} \), we can restate the interference signals correlation matrix as \( R_n = BB^H + \sigma_n^2 I \).

The received signal is processed by a beamformer, in order to enhance the desired signal while mitigating all interference signals:

\[
x = w^H y.
\]

There are several alternatives for designing the beamformer, such as the minimum mean square error (MMSE) criterion [3]. In this paper we are concerned with distortionless array response. In particular, two alternative beamformer criteria are considered, namely the MVDR
and the MPDR. A robust version of the MPDR beamformer, the well-known RCB [7], will be detailed and compared in Sec. IV.

The MVDR beamformer seeks for the best filter set \( w_1 \) such that the overall interference signals’ power is suppressed while the desired signal is maintained:

\[
\begin{align*}
    w_1 &= \arg\min \left\{ w^H R_n w \text{ s.t. } w^H d = 1 \right\}.
\end{align*}
\]  

(4)

The well-known solution to this criterion is given by:

\[
\begin{align*}
    w_1 &= \frac{R_n^{-1} d}{d^H R_n^{-1} d}.
\end{align*}
\]  

(5)

The MPDR criterion is closely related to the MVDR:

\[
\begin{align*}
    w_2 &= \arg\min \left\{ w^H R_g w \text{ s.t. } w^H d = 1 \right\}
\end{align*}
\]  

with a solution given by:

\[
\begin{align*}
    w_2 &= \frac{R_g^{-1} d}{d^H R_g^{-1} d}.
\end{align*}
\]  

(7)

Ideally, without model-errors, both criteria coincide [3]. In the following sections we analyze the beamformers’ performance under model errors in both the interferences and desired channel steering vectors.

III. THEORETICAL ERROR ANALYSIS

In practical applications, accurate steering vectors or noise correlation matrix are rarely available. This may lead to both poor interference reduction and desired signal distortion, and hence cause performance degradation. Cox [1] compared and evaluated the sensitivity of the MVDR and MPDR beamformers and showed that the former, which is directly using the noise correlation matrix, is more robust to desired signal steering errors. In both beamformers, the analysis was done with no model errors on the correlation matrices.

In our contribution, a specific model for the noise correlation matrix is considered. In this scenario, the noise correlation matrix is based on the estimated channels and consists of both interfering point sources and noncoherent noise sources. We derive now a sensitivity analysis to interference signals steering errors.

Denote the estimation of the desired channel vector by \( \tilde{d} \), and the estimated interference channel as \( \tilde{G} \). \( \sigma_v^2 \) and \( \Lambda \) are assumed to be known. Hence, the estimated noise correlation matrix is given by:

\[
\begin{align*}
    \tilde{R}_n &= \tilde{B} \tilde{B}^H + \sigma_v^2 I.
\end{align*}
\]  

(8)

Note, that while the noise correlation matrix \( R_n \), used by the MVDR beamformer, is assumed to be erroneously estimated, the received signal correlation matrix \( R_g \), used by the MPDR, is assumed to be accurately estimated.

A. Figure-of-Merit

The output power of both beamformers is given by:

\[
\begin{align*}
    \sigma_x^2 &= w^H R_g w = w^H R_n w + w^H R_n w
\end{align*}
\]  

(9)

where \( w \) is either the MVDR filters \( w_1 \) or MPDR filters \( w_2 \).

Following Cox [1], we use the SINR improvement, namely, \( \text{SINR}_{\text{mpdr}} = \frac{\text{SINR}_{\text{mvr}}}{\text{SINR}_{\text{mvr}}^\text{mpdr}} \), as a performance measure comparing both beamformers.

Since the input SINR is independent of the beamformer criterion, the output SINR will be used for the comparison. Define,

\[
\begin{align*}
    \text{SINR}_{\text{out}} &= \frac{w^H R_g w}{w^H R_n w} = \frac{\sigma_x^2}{w^H R_n w}
\end{align*}
\]  

(10)

Note, that \( w \) is calculated using inaccurate estimates of \( d \) and \( R_n \). Hence, the constraint \( w^H d = 1 \) is not met. Therefore, the numerator can be regarded as a distortion term and the denominator is related to the noise level at the output of the beamformer.

B. Error Formulation for MPDR

The received signal correlation matrix \( R_g \) is assumed to be accurately estimated. Hence the expression for the output SINR of the MPDR beamformer, given by Cox [1] remains unaltered:

\[
\begin{align*}
    \text{SINR}_{\text{out}}^{w_2} &= \frac{\gamma^2 \text{SINR}_{\text{max}}}{1 + (2 \text{SINR}_{\text{max}} + \text{SINR}_{\text{max}}^d) (1 - \gamma^2)}
\end{align*}
\]  

(11)

where

\[
\begin{align*}
    \gamma^2 &= \frac{|\tilde{d}^H d|}{|\tilde{d}^H \tilde{R}_n^{-1} \tilde{d}| < |\tilde{d}^H d; \tilde{R}_n^{-1} |}
\end{align*}
\]  

(12)

with \( 0 \leq \gamma^2 \leq 1 \). The weighted inner product is defined as

\[
\begin{align*}
    <a, b, c> = \gamma a^H C b
\end{align*}
\]  

and the maximum SINR obtained in the error-free estimation case is defined as

\[
\begin{align*}
    \text{SINR}_{\text{max}} = \sigma_d^2 <d, d; R_n^{-1}>.
\end{align*}
\]  

C. Error formulation for MVDR

In our scenario the noise correlation matrix is comprised of all interference steering vectors. We derive in the sequel an expression for the output SINR of the MVDR beamformer. The power of the signal component at the output of the MVDR beamformer is given by

\[
\begin{align*}
    \sigma_d^2 |w^H d|^2 &= \sigma_d^2 \left( \frac{\tilde{d}^H \tilde{R}_n^{-1} \tilde{d}}{d^H \tilde{R}_n^{-1} d} \right)^2
\end{align*}
\]  

(13)

\[
\begin{align*}
    &= \sigma_d^2 \left( \frac{d^H \tilde{R}_n^{-1} d}{d^H \tilde{R}_n^{-1} d} \right)^2
\end{align*}
\]  

(14)

and the power of the noise component is given by

\[
\begin{align*}
    w_1^H R_n w_1 &= \frac{d^H \tilde{R}_n^{-1} d}{d^H \tilde{R}_n^{-1} d} = \frac{<\tilde{d}, d; \tilde{R}_n^{-1}>}{<\tilde{d}, d; \tilde{R}_n^{-1}>}
\end{align*}
\]  

(15)

Collecting terms we get

\[
\begin{align*}
    \text{SINR}_{\text{out}}^{w_1} &= \frac{\gamma^2 <d, d; \tilde{R}_n^{-1}>^2}{<d, d; \tilde{R}_n^{-1}>}<d, d; \tilde{R}_n^{-1}>
\end{align*}
\]  

(16)

Using Woodbury identity [8], the inverse of the estimated noise correlation matrix [defined by (8)] is given by

\[
\begin{align*}
    \tilde{R}_n^{-1} &= \frac{1}{\sigma_v^2} \left( I - \tilde{B} \left( \sigma_v^2 I + \tilde{B}^H \tilde{B} \right)^{-1} \tilde{B}^H \right).
\end{align*}
\]  

(17)

Using the last expression and (8) we have

\[
\begin{align*}
    \tilde{R}_n \tilde{R}_n^{-1} &= \frac{1}{\sigma_v^2} \left( \sigma_v^2 I + \tilde{B} \tilde{B} \tilde{B}^H \right)
\end{align*}
\]  

(18)
The last expression is comprised of two terms. Substituting each of the expressions into the denominator of (15) we get (after cumbersome manipulations):

$$\alpha \triangleq \frac{1}{\sigma_d^2} \tilde{d}^H \tilde{R}_n^{-1} \left( \sigma_d^2 I + B B^H \right) \tilde{d} =$$

$$< \tilde{d}, d; \tilde{R}_n^{-1} > + \frac{1}{\sigma_d^2} \sum_{i=1}^{K} < \tilde{d}, b_i; \tilde{R}_n^{-1} > < b_i, \tilde{d}; I >$$

$$\beta \triangleq \frac{1}{\sigma_d^2} \tilde{d}^H \tilde{R}_n^{-1} \left( \sigma_d^2 I + B B^H \right) B$$

$$\times \left( \left( \sigma_d^2 I + B B^H \right)^{-1} B^H \tilde{d} \right)$$

$$= \sum_{i=1}^{K} \left[ \sum_{j=1}^{K} t_{i,j} < \tilde{d}, b_i; \tilde{R}_n^{-1} > < b_j, \tilde{d}; I >$$

$$+ \frac{1}{\sigma_d^2} \sum_{j=1}^{K} < \tilde{d}, b_j; \tilde{R}_n^{-1} > < b_j, b_i; I >$$

$$\times \sum_{j=1}^{K} t_{i,j} < b_j, \tilde{d}; I > \right]$$

(20)

where the terms $t_{i,j}$ are the components of the matrix $T$ defined as

$$T = \left( \sigma_d^2 I + B B^H \right)^{-1}.$$  (21)

Collecting terms using (15), (18) and (19), we finally get the expression for the SINR at the output of the MVDR beamformer:

$$\text{SINR}_{\text{out}} = \frac{\sigma_d^2 < \tilde{d}, d; \tilde{R}_n^{-1} >^2}{\alpha - \beta}. \quad (22)$$

IV. PERFORMANCE EVALUATION

This section is dedicated to performance evaluation of the MVDR and MPDR beamformers. The RCB [7], which is a variant of the MPDR beamformer, robust against erroneous desired signal steering vector, is used for comparison.

A. Test procedure

We begin the evaluation by examining the ratio of the SINR improvement [defined by (10)] between the MPDR (11) and the MVDR (15). We then compare the SINR gain of the MVDR and MPDR beamformers with that of the RCB, proposed by Stoica et al. [7]. The RCB is an extension of the standard Capon beamformer (i.e. MPDR) to the case of uncertainty in the steering vectors. Leshem and Gamot [9] used the RCB for successive interference cancelation in the MIMO communications framework. The RCB minimizes the following criterion:

$$\tilde{d} = \text{argmin} \left\{ d^H R_y^{-1} d \text{ s.t. } \| d - \tilde{d} \|^2 = \epsilon \right\}. \quad (23)$$

where $\tilde{d}$ is the initial estimate of the steering vector. The solution for $\tilde{d}$ is given by:

$$\tilde{d} = \tilde{d} - (I + \lambda R_y)^{-1} \tilde{d}. \quad (24)$$

The Lagrange multiplier $\lambda$ is obtained by solving

$$g(\lambda) = \| (I + \lambda R_y)^{-1} \tilde{d} \|^2 = \epsilon. \quad (25)$$

Solving this equation necessitates knowing the uncertainty parameter $\epsilon$, which is unavailable in many cases. Although unavailable, we assume, for simplicity that $\epsilon$ is known. Finally, we obtain the robust beamformer by substituting in the standard beamformer the initial steering vector with the updated estimation:

$$\mathbf{w}_{\text{RCB}} = \frac{R_y^{-1} \tilde{d}}{d^H R_y^{-1} d}. \quad (26)$$

Simulations were carried out using 4 signals impinging on an array of 5 antenna elements from directions $\theta = \{ 0^\circ, 40^\circ, 80^\circ, 110^\circ \}$. All signals are equi-power and the spatial white noise power $\sigma_d^2$ is set in the range $-10\text{dB}$ to $20\text{dB}$. The erroneous steering angle of all signals (either desired or interference) was set to $\Theta_i = \Theta_0 + \Delta \Theta_i, i = 1, 2, 3, 4$. The steering error is assumed to be Gaussian distributed $\Delta \Theta \sim \mathcal{N}(\eta, (0.2\eta)^2)$, where $\eta > 0$. We repeated each experiment 100 times and averaged the results.

B. Results

Fig. 1 depicts the ratio of the SINR improvements of the MVDR and MPDR when the noise correlation matrix is erroneously estimated, due to steering errors. It can be seen that in spatial white noise regime (SNR = $-10\text{dB}$), the differences between MVDR and MPDR are negligible. However, in the spatial interference regime (SNR > $0\text{dB}$), the SINR improvement ratio between the beamformers is significantly higher. To gain some insight on the SINR improvement for various values of SNR and MPDR when the noise correlation matrix is erroneously estimated, due to steering errors. It can be seen that in spatial white noise regime (SNR = $-10\text{dB}$), the differences between MVDR and MPDR are negligible. However, in the spatial interference regime (SNR > $0\text{dB}$), the SINR improvement ratio between the beamformers is significantly higher. To gain some insight on the SINR improvement for various values of SNR for MPDR, MVDR beamformer in comparison with RCB. The improvement as a function of $\Delta \Theta$ is depicted in Fig. 2(a) for SNR = $-10\text{dB}$ and in Fig. 2(b) for SNR = $20\text{dB}$. Table I provides a comparison of the SINR improvement for various values of SNR for $\eta = 6^\circ$. As stated above, in the spatial interference regime, there are significant differences between MPDR and MVDR. It is also evident that the MVDR outperforms the RCB, and that the differences gets larger as the estimation error in the steering angle increases.

C. Discussion

As presented above, the performance differences between the MPDR, MVDR and RCB are insignificant in the white noise regime. This can be attributed to the tendency of all beamformers to converge to the delay and sum beamformer when the white noise level is high. In the spatial interference regime, the MPDR experiences
a considerable performance degradation, which increases with the steering error. This can be explained by examining the beampattern depicted in Fig. 3 (in the range $\pm 60^\circ$) for the case of $\eta = 10^\circ$ and SNR = 20dB. Both beamformers are erroneously steered towards the desired source. The MPDR, which utilizes the received signal correlation matrix in the minimization criterion, considers the actual desired source direction as an interference, and therefore aims at directing null in its angle. This phenomenon dramatically decreases the output SINR. The RCB partially compensates for the steering errors (by broadening the main lobe), resulting in less desired signal suppression. The MVDR which only uses the noise component in the minimization criterion does not direct a null towards the desired source and therefore exhibits an increased robustness to steering errors.

V. CONCLUSION

In this paper, a sensitivity analysis of distortionless beamformer was performed. The case of multiple interference signals with erroneous steering directions was analyzed. An expression for the output SINR obtained by the MVDR and MPDR beamformers was derived and experimentally evaluated. Furthermore, the SINR improvements of the two designated beamformers was compared with SINR improvement of the RCB, a well-known robust variant of the MPDR beamformer. It is shown that the MVDR which utilizes the noise correlation matrix $R_n$ in its minimization criterion, outperforms the MPDR and RCB, both using the received signal correlation matrix $R_y$, although the latter exhibits an improved robustness to steering errors. A fundamental conclusion is that in scenarios in which $R_n$ is available, it is advisable to use the MVDR beamformer, which shows significant robustness to errors in steering vectors estimation. In modern communication systems this scenario is often encountered. We stress, that in applications for which $R_n$ is unavailable, the RCB should be used instead of the MVDR beamformer. An ongoing research extends the results of this paper to fading rather than line of sight channels.

REFERENCES


Fig. 2. SINR improvement as a function of $\eta$ for MPDR, MVDR and RCB.

![Fig. 2. SINR improvement as a function of $\eta$ for MPDR, MVDR and RCB.](image)

(b) SNR = 20dB

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<th>RCB</th>
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TABLE I

SINR IMPROVEMENT FOR VARIOUS VALUES OF SNR FOR MVDR, MPDR AND RCB. STEERING ERROR MEAN $\eta = 6^\circ$. ALL VALUES ARE IN DB.

![Fig. 3. Beampattern for ideal MVDR beamformer with exact steering vector, MVDR, MPDR and RCB beamformers in high SNR. Steering error mean $\eta = 10^\circ$.](image)

Fig. 3. Beampattern for ideal MVDR beamformer with exact steering vector, MVDR, MPDR and RCB beamformers in high SNR. Steering error mean $\eta = 10^\circ$. 

![Image of beampattern for ideal MVDR beamformer with exact steering vector, MVDR, MPDR and RCB beamformers in high SNR. Steering error mean $\eta = 10^\circ$.](image)


