Bargaining Solution for Partial Orthogonal Transmission over Frequency Selective Interference Channel

Ephraim Zehavi\textsuperscript{1} (Senior member) and Amir Leshem\textsuperscript{1} (Senior member)

Abstract—In a wireless OFDM communication system multiple players share the same spectrum while experiencing different channel realizations. Allocation algorithms exploit this feature and optimize the allocation of frequency bins according to various optimization criteria (e.g., weighted Max-Min, Nash Bargaining Solution (NBS), etc). These allocations do not allow players to transmit on the same frequency bin simultaneously. Thus, the communication channel is not fully utilized when the mutual interferences are weak. Here we propose a more general allocation scheme, called Partial Orthogonal Allocation (POA), where players can transmit simultaneously on some of the frequency bins, and maintain orthogonality of transmission on the other frequency bins. For this schema, the Nash Bargaining Solution is proposed and analyzed.

Keywords: Spectrum optimization, distributed coordination, game theory, Nash bargaining solution, interference channel, multiple access channel.

I. INTRODUCTION

Radio resource allocation is an essential component of optimizing the utility of radio services. Wireless services providers suffer from a limited spectrum and the high cost of infrastructure, whereas customers want high data rate services at low cost.

In recent years cooperative approaches derived from game theory have been used for efficient radio resource allocation. The most popular approach is the Nash Bargaining Solution (NBS) \cite{1}, \cite{2}. The NBS is based on four axioms that lead to a unique solution for the bargaining problem \cite{3}. The NBS was applied to a flat fading channel (SISO \cite{4}), MISO \cite{5} and MIMO, as well as to a frequency selective channel under two types of constraints: an average power constraint \cite{1}, \cite{6} and a power mask constraint \cite{2}. Other bargaining solutions, under different axioms have also been proposed. The Generalized Nash Bargaining Solution (GNBS) \cite{7}, and the Kalai-Smorodinsky solution \cite{8} were recently applied to radio resource allocation in \cite{9}, \cite{10}. Most of the works in this area have concentrated on a cooperative game corresponding to the joint FDM/TDM achievable rate region where players do not transmit on the same frequency band simultaneously. In this paper we extend the achievable rate region by allowing simultaneous transmission by all players, a fraction of the time, on some of the frequency bins. We call this scheme Partial Orthogonal Allocation (POA). This extension allows us to find the NBS that outperforms previous allocations based on the joint FDM/TDM approach.

The structure of the paper is as follows: In section II we describe the basic models of interference channels, and define the achievable rate region. We also present the Partial Orthogonal FDM/TDM game and the Nash Bargaining solution to this game. Section III provides a detailed solution for the 2-player game followed by an example. We end with conclusions and a summary.

II. GAME OVER FREQUENCY SELECTIVE CHANNELS UNDER MASK CONSTRAINT

In this section we define a new cooperative game corresponding to the Partial Orthogonal FDM/TDM achievable rate region for the frequency selective \( N \) player interference channel under a PSD mask constraint. Let the \( K \) channel matrices at frequencies \( k = 1, \ldots, K \) be given by \((H_k : k = 1, \ldots, K)\). Each player is allowed to transmit in the \( k \)’th frequency bin with a maximum power \( p(k) \). In the non-cooperative scenario, under mask constraint, all players transmit at the maximal power they can use. Thus, all players choose the PSD, \( p = \langle p_n(k) : 1 \leq k \leq K \rangle \). The payoff for player \( n \) in the non-cooperative game is given by:

\[
R_nC(p_n) = \sum_{k=1}^{K} R_{nC}(k),
\]

where

\[
R_{nC}(k) = \log_2 \left( 1 + \frac{\mathbb{E}_{\sigma_n} \left( h_{nm}(k) \right) \sigma_n(k) }{\mathbb{E}_{\sigma_n} \left( w_{nm}(k) + \sigma_n(k) \right) } \right).
\]

Here, \( R_{nC} \) is the capacity available to player \( n \) under a PSD mask constraint distributions \( p \). \( \sigma_n^2(k) > 0 \) is the noise present at the \( n \)’th receiver at frequency bin \( k \). In order to simplify the notation, we assume that the width of each bin is normalized to 1 without loss of generality.

We now define the cooperative game \( G_{OTF}(N,K,p) \).

Definition 2.1: The Partial Orthogonal FDM/TDM game \( G_{OTF}(N,K,p) \) is a game between \( N \) players transmitting over \( K \) frequency bins under a common PSD mask constraint. Each player has full knowledge of the channel matrices \( H_k \). The following conditions hold:

1) Player \( n \) transmits using a PSD limited by \( \langle p_n(k) : k = 1, \ldots, K \rangle \).

2) Strategies for player \( n \) are vectors \( \beta = [\beta(1) , \ldots, \beta(K)]^T \), and \( \alpha_n = [\alpha_n(1), \ldots, \alpha_n(K)]^T \).

3) \( \beta(k) \) is the time fraction that frequency bin \( k \) is used by all players. The effective rate on frequency bin \( k \) for player \( n \) during this time is \( R_{nC}(k) \).

4) \( \alpha_n(k) \) is the time fraction that player \( n \) uses the \( k \)’th frequency bin with no interference.
The utility of the $n$’th player is given by
\[ R_n (\alpha_n, \beta) = \sum_{k=1}^{K} \alpha_n(k) R_n (k) + \beta (k) R_{nC} (k), \quad (3) \]
where,
\[ R_n (k) = \log_2 \left( 1 + \frac{|h_n(k)|^2 P_{\text{max}}(k)}{\sigma_n^2 (k)} \right), \quad (4) \]
Note that mutual interference on frequency bin $k$ is avoided by time sharing only for a time fraction of $1 - \beta (k)$. In time fraction $\beta (k)$ all players are transmit on frequency bin $k$. This is a suboptimal strategy since, it would probably be better if only a subset of the players with very low mutual interference to be allowed to transmit simultaneously. However, this assumption makes the optimization problem tractable.

The rate vectors of interest are only the rate vectors that dominate component-wise the rates that each player can achieve, independently of the other players’ coding strategy. The best rate pairs that can be achieved independently of the other players’ strategies form a Nash equilibrium [11]. This implies that all the rates are indeed achievable from a game theoretic perspective. Hence, below we define the game theoretic rate region.

**Definition 2.2:** Let $\mathcal{R}$ be an achievable information theoretic rate region. The game theoretic rate region $\mathcal{R}^G$ is given by
\[ \mathcal{R}^G = \{ (R_1, ..., R_N) \in \mathcal{R} : R_{nC} (p_n) \leq R_n (\alpha_n, \beta), n = 1, ..., N \}, \quad (5) \]
where $R_{nC} (p_n)$ is the rate achievable by player $n$ in a non-cooperative interference game, and $R_n (\alpha_n, \beta)$ is given by equations (3), and (4).

The game theoretic rate region $\mathcal{R}^G$ is a convex set since rate vectors are linearly related to $\alpha$, and $\beta$. To see which pair rates can be achieved by negotiation and cooperation among players we resort to a well known solution termed the Nash Bargaining Solution (NBS). In his seminal papers, Nash proposed four axioms [12], [3] that any solution to the bargaining problem should satisfy. He then proved that there exists a unique solution satisfying these axioms. We now analyze derive NBS to our game, $G_{OTF} (N, K, p)$, and show that there exists a unique point on the boundary of the capacity region which is the solution to the bargaining problem as posed by Nash.

The Nash bargaining can be posed as an optimization problem
\[
\begin{align*}
\text{max} & \quad \prod_{n=1}^{N} (R_n (\alpha_n, \beta) - R_{nC}), \\
\text{subject to:} & \quad \sum_{n=1}^{N} \alpha_n(k) + \beta (k) = 1, \alpha_n(k) \geq 0, \quad \forall n, k \\
& \quad 0 \leq \beta (k), \forall k, \\
& \quad R_{nC} \leq R_n (\alpha_n, \beta), \quad \forall n. 
\end{align*}
\]
This problem is convex and therefore can be solved efficiently using convex optimization techniques. To that end we explore the KKT conditions for the problem. The Lagrangian of the problem $f (\alpha, \beta)$ is given by
\[
f (\alpha, \beta) = -\sum_{n=1}^{N} \log (R_n (\alpha_n, \beta) - R_{nC}) \\
+ \sum_{k=1}^{K} \lambda_k \left( \sum_{n=1}^{N} \alpha_n(k) + \beta (k) - 1 \right) \\
- \sum_{n=1}^{N} \mu_n(k) \alpha_n(k) - \sum_{k=1}^{K} \mu_n^C (k) \beta (k) \\
- \sum_{n=1}^{N} \delta_n (R_n (\alpha_n, \beta) - R_{nC}).
\]
Taking the derivatives with respect to the variables $\alpha_n(k)$, and $\beta (k)$, and comparing the result to zero, we get
\[
\begin{align*}
\frac{R_n (k)}{R_n (\alpha_n, \beta) - R_{nC}} &= \lambda_k - \mu_n(k) - \delta_n R_n (k) \\
\sum_{n=1}^{N} \frac{R_n (k)}{R_n (\alpha_n, \beta) - R_{nC}} &= \lambda_k - \mu_n^C - \sum_{n=1}^{N} \delta_n R_{nC} (k), 
\end{align*}
\]
with the constraints
\[
\begin{align*}
\sum_{n=1}^{N} \alpha_n(k) + \beta (k) &= 1, \\
\delta_n (R_n (\alpha_n, \beta) - R_{nC}) &= 0, \\
\mu_n(k) \alpha_n(k) &= 0, \\
\lambda_k \geq 0, \mu_n^C (k) \beta (k) &= 0.
\end{align*}
\]
Based on (8, 9) one can easily come to the following conclusions:

1) If there is a feasible solution, such that for some $k$, $\beta (k) < 1$, then for all $n$, $\delta_n = 0$.
2) Assume that a feasible solution exists. Then for all players sharing the frequency bin $k$ ($\alpha_n(k) > 0$) we have $\mu_n(k) = 0$, and
\[
\frac{R_n (k)}{R_n (\alpha_n, \beta) - R_{nC}} = \lambda_k, \forall k \text{ satisfying } \alpha_n(k) > 0.
\]
3) For all players who are not sharing the frequency bin $k$, ($\alpha_n(k) = 0$), $\mu_n(k) \geq 0$. Therefore,
\[
\frac{R_n (k)}{R_n (\alpha_n, \beta) - R_{nC}} \leq \lambda_k, \forall k \text{ with } \alpha_n(k) = 0.
\]
4) If only a single player transmits on frequency bin $k$ at any time, i.e. $\beta (k) = 0$ then $\mu_n^C \geq 0$ and
\[
\sum_{n=1}^{N} \frac{R_{nC} (k)}{R_n (\alpha_n, \beta) - R_{nC}} \leq \lambda_k.
\]
5) If all the players transmit simultaneously on the same frequency bin $k$ fraction of the time, i.e. $0 < \beta (k) \leq 1$ then $\mu_n^C = 0$ and
\[
\sum_{n=1}^{N} \frac{R_{nC} (k)}{R_n (\alpha_n, \beta) - R_{nC}} = \lambda_k.
\]
The conclusions are very interesting. If the proportional sum of the surplus in a given frequency is equal to the threshold, then the players prefer to transmit simultaneously on this frequency bin part of the time. Moreover, if the rates are random variables with a continuous probability density function over the real line, then the probability that equations (10), and (13) will be satisfied simultaneously is zero. Therefore, in this case the set of all frequency bins that are allocated to player $n$ part
of the time, and part of the time to player j, and part of the
time are simultaneously shared by both players is an empty
set.

III. THE TWO PLAYERS GAME

In the two player game, the KKT equations can be simplified
and provide a simple algorithm for computing the NBS. Let
us first rewrite the equations in this case. The first equation
provides a rule for partition the bins between the players.
Assume that a feasible solution exists, and the two players
share the frequency bin k (0 < α(1)(k), α(2)(k) < 1), and
\[ \frac{R_1(k)}{R_1(\alpha_1, \beta) - R_{1C}} = \frac{R_2(k)}{R_2(\alpha_2, \beta) - R_{2C}} = \lambda_k. \]  
(14)
and for all other frequency bins we have the rule,
\[ \frac{R_1(k')}{R_1(\alpha_1, \beta) - R_{1C}} < \frac{R_2(k')}{R_2(\alpha_2, \beta) - R_{2C}}. \] 
(15)
Explicitly, if a frequency bin is shared between the players it
is the bin for which the ratio satisfies
\[ \lambda^* = \frac{R_1(\alpha_1, \beta) - R_{1C}}{R_2(\alpha_2, \beta) - R_{2C}} = \frac{R_1(k_s)}{R_2(k_s)}. \]  
(16)
If none of the frequency bins are shared by the two players
then there is frequency bin k for which
\[ \frac{R_1(k_s - 1)}{R_2(k_s - 1)} > \lambda^* = \frac{R_1(\alpha_1, \beta) - R_{1C}}{R_2(\alpha_2, \beta) - R_{2C}} > \frac{R_1(k_s)}{R_2(k_s)}. \]  
(17)
The frequency bins in which both players prefer to transmit
all the time must satisfy equations (10-13). Using simple mathema-
tical manipulation it easy to show that these frequency bins
must satisfy the conditions
\[ \frac{R_1(k)}{R_1(\alpha_1, \beta) - R_{1C}} \leq \frac{R_{1C}(k)}{R_2(\alpha_1, \beta) - R_{1C}} + \frac{R_{2C}(k)}{R_2(\alpha_2, \beta) - R_{2C}}, k \leq k_s \]
\[ \frac{R_1(k)}{R_2(\alpha_1, \beta) - R_{1C}} \leq \frac{R_{1C}(k)}{R_1(\alpha_1, \beta) - R_{1C}} + \frac{R_{2C}(k)}{R_2(\alpha_2, \beta) - R_{2C}}, k \geq k_s. \]  
(18)
These conditions can be written as
\[ \frac{R_1(k) - R_{1C}(k)}{R_2(\alpha_2, \beta) - R_{2C}} \leq \lambda^*, k \leq k_s \]
\[ \frac{R_1(k) - R_{1C}(k)}{R_2(\alpha_2, \beta) - R_{2C}} \geq \lambda^*, k \geq k_s. \]  
(19)
The interpretation is that a frequency bin k is allocated
solely to player 1 (or 2) only if the ratio of the rates on
\[ R_1(k)/R_2(k) \] is above (below) the threshold \( \lambda^* \), and the ratio
between the surplus (extra rate above the competition) rate of
player 1 (or 2) vs. the competition rate of player 2, (or 1), is
below (above) the threshold \( \lambda^* \).
Let first assume that there is a solution for NBS. Then, the
threshold \( \lambda^* \) induces a partition of the frequency bins into
seven sets as follows:
• Sets \( I_n \)-All frequency bins that can only be used by a
single player \( n = 1, 2 \), i.e.
\[ I_1 = \{ k : \frac{R_1(k)}{R_{1C}(k)} \geq \lambda^* \text{ and } \frac{R_1(k) - R_{1C}(k)}{R_2(k)} \geq \lambda^* \} \] 
(20)
\[ I_2 = \{ k : \frac{R_1(k)}{R_{1C}(k)} < \lambda^* \text{ and } \frac{R_1(k) - R_{1C}(k)}{R_2(k)} < \lambda^* \} \]
• Set \( I_c \)-All frequency bins that are used by both players
simultaneously, i.e.
\[ I_c = \{ k : \frac{R_1(k)}{R_{1C}(k)} \geq \lambda^* \text{ and } \frac{R_1(k) - R_{1C}(k)}{R_{2C}(k)} < \lambda^* \} \]
\[ \cup \{ k : \frac{R_1(k)}{R_{1C}(k)} \leq \lambda^* \text{ and } \frac{R_1(k) - R_{1C}(k)}{R_{2C}(k)} > \lambda^* \} \]  
(21)
• Set \( I_{nc} \)-All frequency bins that are allocated part of the
time to player n and part of the time to both players.
\[ I_{1c} = \{ k : \frac{R_1(k)}{R_{1C}(k)} > \lambda^* \text{ and } \frac{R_1(k) - R_{1C}(k)}{R_{2C}(k)} = \lambda^* \} \]
\[ I_{2c} = \{ k : \frac{R_1(k)}{R_{1C}(k)} < \lambda^* \text{ and } \frac{R_1(k) - R_{1C}(k)}{R_{2C}(k)} = \lambda^* \} \]  
(22)
• Set \( I_{nj} \)-All frequency bins that are allocated part of the
time to player n and player j and part of the time to both
players allocated by both players.
\[ I_{njc} = \{ k : \frac{R_1(k)}{R_{1C}(k)} = \lambda^* \text{ and } \frac{R_1(k) - R_{1C}(k)}{R_{2C}(k)} = \lambda^* \} \]
\[ \cup \{ k : \frac{R_1(k)}{R_{1C}(k)} = \lambda^* \text{ and } \frac{R_1(k) - R_{1C}(k)}{R_{2C}(k)} < \lambda^* \} \]  
(23)
• Set \( I_{nj} \)-All frequency bins that are allocated part of the
time to player n and player j and never used by the two
players simultaneously.
\[ I_{nj} = \{ k : \frac{R_1(k)}{R_{1C}(k)} = \lambda^* \text{ and } \frac{R_1(k) - R_{1C}(k)}{R_{2C}(k)} > \lambda^* \} \]
\[ \cup \{ k : \frac{R_1(k)}{R_{1C}(k)} = \lambda^* \text{ and } \frac{R_1(k) - R_{1C}(k)}{R_{2C}(k)} < \lambda^* \} \]  
(24)
We now can write the total rates for each of the players as
follows
\[ R_1(\alpha_1, \beta) = \sum_{k \in I_1} R_1(k) + \sum_{k \in I_2, I_{1c}, I_{2c}, I_{nj}, I_{njc}} R_1(k) \]
\[ + \sum_{k \in I_{1c}} \alpha_1(k) (R_1(k) - R_{1c}(k)) \]
\[ + \sum_{k \in I_{nj}} \alpha_1(k) R_1(k) \]
\[ - \sum_{k \in I_{2c}} I_{njc} \alpha_2(k) R_1(k) \]  
(25)
\[ R_2(\alpha_1, \beta) = \sum_{k \in I_2} R_2(k) + \sum_{k \in I_1, I_{1c}, I_{2c}, I_{nj}, I_{njc}} R_2(k) \]
\[ + \sum_{k \in I_{1c}} \alpha_2(k) R_2(k) \]
\[ + \sum_{k \in I_{nj}} \alpha_2(k) R_2(k) \]
\[ - \sum_{k \in I_{2c}} I_{njc} \alpha_1(k) R_2(k) \]  
(26)
After tedious but simple mathematical manipulations we obtain that \( \lambda^* \) is the solution to the equation
\[ \frac{R_1(\alpha_1, \beta) - R_{1C}}{R_2(\alpha_2, \beta) - R_{2C}} = \frac{A_{\lambda^*} - X_{\lambda^*}}{B_{\lambda^*} + X_{\lambda^*}} = \lambda^*, \]  
(27)
where, \( A_{\lambda^*} \), \( B_{\lambda^*} \) and \( X_{\lambda^*} \) are given by
\[ A_{\lambda^*} = \sum_{k \in I_1 \cup I_{nj}} R_1(k) - \sum_{k \in I_1 \cup I_{2c} \cup I_{nj}} R_{1c}(k) \]
\[ B_{\lambda^*} = \sum_{k \in I_2} R_2(k) - \sum_{k \in I_1 \cup I_{2c} \cup I_{nj}} R_{2c}(k) \]
\[ X_{\lambda^*} = - \sum_{k \in I_{1c} \cup I_{njc} \cup I_{nj}} \alpha_1(k) R_{1c}(k) \]
\[ + \sum_{k \in I_{nj}} \alpha_2(k) R_2(k) \]
\[ + \sum_{k \in I_{2c} \cup I_{njc}} \alpha_2(k) (R_2(k) - R_{2c}(k)). \]  
(28)
A_\lambda (B_\lambda) is an increasing (decreasing) function of \lambda, and X_\lambda is bounded according to

\[
X_{\lambda, \min} \leq X(\lambda) \leq X_{\lambda, \max} \\
X_{\lambda, \min} = -\sum_{k \in I_{1c} \cup I_{n_{jc}}} R_{2c}(k) \\
X_{\lambda, \max} = \sum_{k \in I_{nc} \cup I_{n_{jc}}} R_{2c}(k) + \sum_{k \in I_{2c} \cup I_{n_{jc}}} R_{2}(k) - R_{2c}(k) \\
\tag{29}
\]

Let us denote by \Lambda and \hat{\Lambda} the following sets

\[
\Lambda = \left\{ \Lambda(k) = \frac{R_{2c}(k)}{R_{2c}(k) - R_{2}(k)} | k = 1, \ldots, K \right\} \\
\hat{\Lambda} = \left\{ (\Lambda(i + 1) + \Lambda(i)) / 2, i = 1, \ldots, 3K - 1 \right\} \\
\tag{30}
\]

The value of \lambda can be found by a simple search in the interval \(\min \lambda \in \Lambda, \max \lambda \in \hat{\Lambda}\). If \lambda^* \in \Lambda; then at most one \alpha is in the interval \(0 < \alpha < 1\) to satisfy the equality \(X = \frac{1}{2} (\frac{\lambda^*}{\lambda} - B^{(2)}_\lambda)\). There area total of 3K checks to be done. A valid solution has to satisfy the constraints:

\[
A_{\lambda} \geq \lambda X_{\lambda} \\
B_{\lambda} \geq -X_{\lambda} \\
X_{\lambda} \geq -\sum_{k \in I_{1c} \cup I_{n_{jc}}} R_{2c}(k) \\
X_{\lambda} \leq \sum_{k \in I_{nc} \cup I_{n_{jc}}} R_{2}(k) + \sum_{k \in I_{2c} \cup I_{n_{jc}}} R_{2}(k) - R_{2c}(k) \\
\tag{31}
\]

If there is no such solution, then \lambda^* \notin \Lambda and the sets \(I_{1c}, I_{2c}, I_{nc}, I_{n_{jc}}\) are empty (i.e., \(X_{\lambda} = 0\)). Thus, we can arbitrarily search over the set \hat{\Lambda} to find the threshold that maximizes the product \(A_{\lambda} \bullet B_{\lambda}\). The complexity of the algorithm is on the order of \(O(K^2)\), since each new threshold induces a new set partition of the frequency bins.

Based on the previous discussion the algorithm is described in Table I. In the first stage, the algorithm builds a table of all the ratios in the set \(\Lambda^* = \Lambda \cup \hat{\Lambda}\) and sorts them in a non-increasing order. In the second stage the algorithm searches for \(\lambda \in \Lambda\) that satisfies the constraints in Eq. (31); if there is no solution, then the algorithm searches over the set \hat{\Lambda}, for \(\lambda\) that maximizes the product \(A_{\lambda} \bullet B_{\lambda}\). For purposes of simplification of the algorithm we introduce this version, although the complexity of the second search can be reduced dramatically.

To demonstrate the algorithm we compute the Nash Bargaining for the following example:

**Example I:** Consider two players who are communicating over a 2x2 Gaussian interference channel with 8 frequency bins. The players’ rates with and without cooperation in each frequency bin are given in Table II. There are 47 possible values of thresholds. Table III shows the partition of the frequency bins as a function of the threshold \(\lambda\). The achievable rate regions for joint FDM/TDM and Partial Orthogonal FDM/TDM are shown in Figure 1. The achievable rate region of Partial Orthogonal FDM/TDM includes the achievable rate region of joint FDM/TDM. The competition point is outside the achievable rate region of joint FDM/TDM and therefore there is no NBS for joint FDM/TDM. However for the Partial Orthogonal FDM/TDM there is a NBS.

<table>
<thead>
<tr>
<th>Initialization: Sort the ratios (\lambda \in \Lambda^* = \Lambda \cup \hat{\Lambda}) in decreasing order.</th>
</tr>
</thead>
<tbody>
<tr>
<td>For (i=1) to (</td>
</tr>
<tr>
<td>Select (\lambda_i \in \Lambda).</td>
</tr>
<tr>
<td>Calculate the values of (A_{\lambda_i}, B_{\lambda_i}).</td>
</tr>
<tr>
<td>and (X_{\lambda_i, \min}, X_{\lambda_i, \max}).</td>
</tr>
<tr>
<td>If (X_{\lambda_i, \min} \leq X_{\lambda_i} \leq 0)</td>
</tr>
<tr>
<td>The optimal threshold is (\lambda_i).</td>
</tr>
<tr>
<td>Set (\alpha_2(k) = 0 \forall k \in I_{2c} \cup I_{n_{jc}} \cup I_{n_{jc}}).</td>
</tr>
<tr>
<td>Assign a value (\alpha_1(k) = 1) to subset of frequency bins in the sets (I_{1c}, I_{n_{jc}},) and assign to one frequency bin in these sets a value of (0 \leq \alpha_1 \leq 1),</td>
</tr>
<tr>
<td>such that (X_{\lambda_i} = -\sum_{k \in I_{1c} \cup I_{n_{jc}}} \alpha_1(k) R_{2c}(k)).</td>
</tr>
<tr>
<td>This is NBS, stop.</td>
</tr>
<tr>
<td>Else if (0 &lt; X_{\lambda_i} \leq X_{\lambda_i, \max}\</td>
</tr>
<tr>
<td>The optimal threshold is (\lambda_i).</td>
</tr>
<tr>
<td>Set (\alpha_1(k) = 0 \forall k \in I_{1c} \cup I_{n_{jc}}).</td>
</tr>
<tr>
<td>Assign a value (\alpha_2(k) = 1) to subset of frequency bins in the sets (I_{1c}, I_{2c}, I_{n_{jc}},) and assign to one frequency bin in these sets a value of (0 \leq \alpha_2 \leq 1),</td>
</tr>
<tr>
<td>such that (X_{\lambda_i} = \sum_{k \in I_{1c}} \alpha_2(k) R_{2c}(k) + \sum_{k \in I_{2c} \cup I_{n_{jc}}} \alpha_2(k) (R_{2}(k) - R_{2c}(k))).</td>
</tr>
<tr>
<td>This is NBS, stop.</td>
</tr>
<tr>
<td>End</td>
</tr>
</tbody>
</table>

| End |

IV. Conclusion

In this paper we defined a new game we dubbed the Partial Orthogonal FDM/TDM, and found the NBS to this game. The new game always has a NBS, and when the mutual interference becomes weaker the difference between the Nash equilibrium point and NBS decreases. In the extreme case where the mutual interference approaches zero, the Nash equilibrium point lies on the boundaries of the game rate region \(R^G\). This model for 2-players effectively overcomes the case of weak mutual interference, and there is simple algorithm for finding NBS. For the case of N-players the model extends the achievable rate region. The game rate region \(R^G\) can also be used to improve other solutions, like the Generalized Nash Bargaining Solution (GNBS) [7], [9], the Kalai-Smorodinsky solution [8], [9], and the weight Max-Min [13].

REFERENCES

TABLE II
PLAYER RATES IN EACH FREQUENCY BIN AND RATIOS

<table>
<thead>
<tr>
<th>Bin k</th>
<th>( R_1 )</th>
<th>( R_{1c} )</th>
<th>( R_2 )</th>
<th>( R_{2c} )</th>
<th>( \frac{R_1(k)}{R_{1c}(k)} )</th>
<th>( \frac{R_2(k)}{R_{2c}(k)} )</th>
<th>( R_{1c}(k) - R_{2c}(k) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>95.4</td>
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TABLE III
THE PARTITIONS OF BINS INTO SETS FOR SEVERAL THRESHOLD VALUES.

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Fig. 1. Achievable rate region for joint FDM/TDM (solid line) and Partial Orthogonal FDM/TDM (broken line), and the new NBS point.