An Extension of the Nemhauser & Trotter Theorem to Generalized Vertex Cover with Applications

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Minimum Vertex Cover

Coffee Chain Problem:
- Place coffee house in every street segment
- Coffee houses are placed in street corners
- Locations have different costs
- Minimize costs
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Vertex Cover:

Instance: Graph $G = (V, E)$, weights $w : V \rightarrow \mathbb{R}^+$
Solution: $C \subseteq V$ such that $\forall e \in E, \ e \cap C \neq \emptyset$
Objective: $\min \sum_{u \in C} w(u)$
Half Integrality of LP Formulation

\[
\begin{align*}
\text{min} & \quad \sum_u w(u)x(u) \\
\text{s.t.} & \quad x(v) + x(u) \geq 1 \quad \forall (v,u) \in E \\
& \quad x(v) \geq 0 \quad \forall v \in V
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**Half Integrality:**

- $x$ is basic feasible $\Rightarrow x \in \{0, \frac{1}{2}, 1\}^n$
- $\exists$ optimal solution $x^* \in \{0, \frac{1}{2}, 1\}^n$
Minimizing $\sum w(v)x(v)$ subject to $x(v) + x(u) \geq 1 \quad \forall (v, u) \in E$

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Computing Half-Integral Optimal Solution:
Using algorithm for Vertex Cover in bipartite graphs

$V_1 = \{v : |\{v, v'\} \cap C| = 2\}$
$V_{1/2} = \{v : |\{v, v'\} \cap C| = 1\}$
$V_0 = \{v : |\{v, v'\} \cap C| = 0\}$
Partition of $V$:

- $V_1 = \{v : x(v) = 1\}$
- $V_{1/2} = \{v : x(v) = \frac{1}{2}\}$
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Nemhauser & Trotter Theorem

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Nemhauser & Trotter Theorem (1975):

(i) $C$ is a vertex cover in $G[V_{1/2}] \Rightarrow w(C) \geq w(V_{1/2})/2$

(ii) $C$ is $\alpha$-approx for $G[V_{1/2}] \Rightarrow V_1 \cup C$ is $\alpha$-approx for $G$
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Corollaries:

- First 2-approximation algorithm for Vertex Cover
- We may assume that $\text{OPT} \geq \frac{w(V)}{2}$ when designing an approximation alg. for Vertex Cover
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Application:
$\exists$ independent set $I$ s.t. $w(I) \geq \varepsilon \cdot w(V)$

$\Rightarrow w(V \setminus I) \leq (1 - \varepsilon)w(V) \leq 2(1 - \varepsilon)\text{OPT}$
Corollary: We may assume that $\text{OPT} \geq \frac{w(V)}{2}$

Application:
\[ \exists \text{ independent set } I \text{ s.t. } w(I) \geq \varepsilon \cdot w(V) \]
\[ \Rightarrow w(V \setminus I) \leq (1 - \varepsilon)w(V) \leq 2(1 - \varepsilon)\text{OPT} \]

Example [Hochbaum 83]:
- Using a $k$-coloring of $G$: $\exists I, w(I) \geq \frac{w(V)}{k} \Rightarrow (2 - \frac{2}{k})$-approx
- $G$ is planar: $\exists I, w(I) \geq \frac{w(V)}{4} \Rightarrow \frac{3}{2}$-approx
- $G$ has bounded degree $d$: $\exists I, w(I) \geq \frac{w(V)}{d} \Rightarrow (2 - \frac{2}{d})$-approx
EPTAS for planar graphs [Baker 94]:

- Partition $V$ into $V_0, \ldots, V_k$ such that $G[V \setminus V_i]$ is $k$-outerplanar

- $k + 1$ candidate solutions:

$$C_i = V_i \cup \text{OPT}_i$$
Applications of the Nemhauser & Trotter Theorem

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- Total weight of candidate solutions:
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  \sum_i w(C_i) = w(V) + \sum_i \text{OPT}_i \\
  \leq 2\text{OPT} + (k + 1)\text{OPT} \\
  = (k + 3)\text{OPT}
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- $(1 + \varepsilon)$-approx:

$$\min_i w(C_i) \leq (1 + \frac{2}{k+1})\text{OPT} = (1 + \varepsilon)\text{OPT}$$
Applications of the Nemhauser & Trotter Theorem

Other Applications:

□ (2 − \frac{\log \log n}{2 \log n})-approx [Bar-Yehuda Even 85]

– Short odd cycles are removed using local ratio
– Nemhauser & Trotter Theorem
– Compute vertex cover \( C \) such that \( w(C) \leq (1 - \frac{1}{2k})w(V_{1/2}) \)

□ 2k kernel for Vertex Cover [Chen et al. 01]

– If \( w(V_{1/2}) > 2k \), then \( \text{OPT} > k \)
Generalized Coffee Chain Problem:

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Solution: $C \subseteq V$
Objective: $\min \sum_{u \in C} w(u) + \sum_{e : e \cap C = \emptyset} w(e)$
Half Integrality of LP Formulation

LP Relaxation: min $\sum_u w(v)x(v) + \sum_e w(e)z(e)$

s.t. $x(v) + x(u) + z(e) \geq 1$ $\forall e = (v, u) \in E$

$x(v) \geq 0$ $\forall v \in V$

$z(e) \geq 0$ $\forall e \in E$
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Cost of a Solution:

- Any vertex set \( C \) is a solution
- \( \text{COST}(C) = w(C) + w(V \setminus C, V \setminus C) \)
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Previous Results:
- 2-approx and half integrality [Hochbaum 02]
- Linear time 2-approx [Bar-Yehuda Rawitz 05]
- 2-approx for a more general problem [Hassin Levin 06]
Extended Nemhauser & Trotter Theorem:
Let \((G, w)\) be an instance of \textit{Generalized Vertex Cover}.

There is a poly-time algorithm that partitions \(V\) into 3 subsets, \(V_1, V_0,\) and \(V_{1/2}\), and constructs a weight function \(\tilde{w}\), s.t.

(i) \(C\) is \(\alpha\)-approx for \(G[V_{1/2}], \tilde{w}\) \(\Rightarrow\) \(V_1 \cup C\) is \(\alpha\)-approx for \(G, w\)

(ii) \(C \subseteq V_{1/2}\) \(\Rightarrow\) \(\text{COST}(C) \geq \tilde{w}(V_{1/2})/2\)
Main Result

Extended Nemhauser & Trotter Theorem:
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Corollary:
We may assume that \(\text{OPT} \geq \frac{w(V)}{2}\) when designing an approximation alg. for Generalized VC
More Results:

- $(2 - \frac{2}{d})$-approx for graphs of bounded degree $d$
  
  - Generalized VC can be solved in poly-time time if $d \leq 2$
Applications of the Extended N&T Theorem

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- EPTAS for planar graphs
  - Generalized VC can be solved in \( 2^{O(w)} n \) if \( \text{treewidth}(G) \leq w \)
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- $(2 - \frac{\log \log n}{2 \log n})$-approx
  - Short odd cycles are removed using local ratio
  - “Reduction” to Vertex Cover
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- \((2 − \log \log n - \log n)\)-approx
  - Short odd cycles are removed using local ratio
  - “Reduction” to Vertex Cover

- 2\(k\) kernel for parameterized Generalized VC
  - Parameter is the cost of the solution
Algorithm for Bipartite Graphs:

- Construct network
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- Compute minimum \( s, t \)-cut \( (S, T) \)
Algorithm for Bipartite Graphs:

- Construct network
- Compute minimum $s, t$-cut $(S, T)$
- Return $(L \cap T) \cup (R \cap S)$
Computing Half-Integral Optimal Solution:

- Compute partition of $V$ using algorithm for bipartite graphs:
  
  $V_1 = \{v : |\{v, v'\} \cap C| = 2\}$
  
  $V_{1/2} = \{v : |\{v, v'\} \cap C| = 1\}$
  
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- Observations:
  - $V_0$ is not an independent set
  - There are edges between $V_0$ and $V_{1/2}$
On Proving the Extended N&T Theorem

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We can still show that:
- There exists an optimal solution \( C \) s.t. \( V_1 \subseteq C \subseteq V_1 \cup V_{1/2} \)
- \( C \) is \( \alpha \)-approx for \( G[V_{1/2} \cup V_0] \) \( \Rightarrow \) \( V_1 \cup C \) is \( \alpha \)-approx for \( G \)
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Solution:
- We use local ratio to move some vertices from \( V_{1/2} \) to \( V_1 \)
- New weight function \( \tilde{w} \)
Our Results:

- Extend N&T Theorem to Generalized Vertex Cover
- Applications:
  - \((2 - \frac{1}{d})\)-approx when max degree is \(d\)
  - EPTAS for planar graphs
  - \((2 - \frac{\log \log n}{2 \log n})\)-approx
  - \(2k\) kernal for parametrized variant
Conclusion

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- Extend N&T Theorem to **Generalized Vertex Cover**

- Applications:
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  - EPTAS for planar graphs
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Open Problems:

- Find more applications of extend N&T Theorem
  - Improve approx ratio for **Generalized Vertex Cover**
Conclusion

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- Extend N&T Theorem to **Generalized Vertex Cover**
- Applications:
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  - \(2k\) kernal for parametrized variant

Open Problems:

- Find more applications of extend N&T Theorem
  - Improve approx ratio for **Generalized Vertex Cover**
- Reduction from **Generalized VC** to **Vertex Cover**