Amir Leshem and Ephraim Zehavi

# Game Theory and the Frequency Selective Interference Channel

## A practical and theoretic point of view

The success of unlicensed broadband communication has led to very rapid deployment of communication networks that work independently of each other using a relatively narrow spectrum. For example, the 802.11g standard uses the industrial, scientific, and medical (ISM) band that has a total bandwidth of 80 MHz. This band is divided into 12 partially overlapping bands of 20 MHz. These technologies could become the victims of their own success, since the relatively small number of channels and the massive use of the technology in densely populated metropolitan areas can cause significant mutual interference. This is especially important for high quality real-time distribution of multimedia content that is intolerant to errors as well as latency. Existing 802.11 [wireless fidelity (WiFi)] networks have very limited means to coordinate spectrum with other interfering systems. It would be highly desirable to reduce the interference environment, for instance by distributed spectral coordination between different access points. Another option is centralized access points such as 802.16 [worldwide interoperability for microwave access (WiMax)], where resources are allocated centrally by a single base station. Advanced digital subscriber line (DSL) systems such as ADSL2 and VDSL are facing a similar situation. These systems are currently limited by crosstalk between lines. As such, the DSL environment is another example of a frequency selective


interference channel. While the need to operate over interference limited frequency selective channels is clear in many of the current and future communication technologies, the theoretical situation is much less satisfying. The capacity region of the interference channel is still open (see [1] for a short overview) even for the fixed channel two-user case. Recently, major advances in understanding the situation for flat channels under weak interference have been made. It can be shown that

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in this case, treating the interference as noise leads to a virtually optimal solution. On the other hand, for medium-strength interference as is typical in the wireless environment, the simplest strategy is to use orthogonal signaling, e.g., time division multiple access/frequency division multiple access (TDMA/FDMA) for high spectral efficiency networks, or code division multiple access (CDMA) for very strong interference with low spectral efficiency per user. Moreover, sequential cancellation techniques that are required for the best-known capacity region in the medium-interference case [2] are only practical for small numbers of interferers. The interference channel can be seen as a conflict situation, in which not every achievable rate pair (from an informational point of view) is an acceptable operating point for users. One of the best ways to approach this conflict is to analyze the interference channel using game theoretic tools.

## COMMUNICATION OVER INTERFERENCE LIMITED CHANNELS

The study of game theory for communication networks and power control is a wide field (see [3] for an extensive list of references). There is considerable literature on applying competitive game theory to frequency selective interference channels, starting with early works by Yu et al. [4] and more recently papers by Scutari et al. (see [5] and the references therein). In particular, generalized Nash games have been applied to the weak interference channel [6], and the algorithm in [7] that extends fixed margin iterative waterfilling (FM-IWF) to iterative pricing under a fixed rate constraint.

The fact that competitive strategies can result in significant degradation due to the prisoner's dilemma has been called the price of anarchy [8]. For instance, Laufer and Leshem [9] characterized simple cases of the prisoner's dilemma in interference limited channels. There are two alternatives available to overcome the suboptimality of the competitive approach, namely repeated games or cooperative game theory. Since most works on repeated games have concentrated on flat fading channels, we will focus on cooperative game theoretic approaches. One of the earliest solutions for cooperative games is the Nash bargaining solution (NBS) [10]. Many recent papers have been devoted to analyzing the NBS for the frequency flat interference channel in the single-input, single-output (SISO) [11], [12], multiple-input, single output (MISO) [13], [14], and multiple-input, multiple-output (MIMO) [15] cases. Interesting extensions for log-convex utility functions were described in [16]. Another cooperative model was explored in [17], where cooperation between rational wireless players was studied using coalitional game theory by allowing the receivers to use joint decoding.

In the context of frequency selective interference channels, much less research has been done. Han et al. [18], in a pioneering work, studied Nash bargaining under frequency-division multiplexing/time-division multiplexing (FDM/TDM) strategies and total power constraint. Unfortunately, the algorithms proposed were only suboptimal. Iterative suboptimal algorithms to find a NBS for spectrum allocation under average power constraint were applied in [19]. We have only recently managed to overcome these difficulties by imposing a power spectral density (PSD) mask constraint [20] to obtain computationally efficient solutions to the bargaining problem in the frequency selective SISO and MIMO cases under TDM/FDM strategies. Furthermore, it can be shown [21] that the PSD limited case can be used to derive a computationally efficient converging algorithm in the total power constraint case as well. A very interesting and open problem arises when the users are allowed to treat the interference as noise in some bands, while using orthogonal FDM/TDM strategies in others. This is a very challenging task, since the NBS requires overcoming a highly nonconvex power allocation problem.

As discussed above, the frequency selective interference channel is important, both from a practical as from an information theoretic point of view. We show that it has many intriguing aspects from a game theoretic point of view as well, and that various levels of interference admit different types of game theoretic techniques.

## INTRODUCTION TO INTERFERENCE CHANNELS

Computing the capacity region of the interference channel is an open problem in information theory [22]. A good overview of the results up to 1985 can be found in van der Meulen [1] and the references therein. The capacity region of a general interference channel is not yet known. However, in the last 45 years of research, some progress has been made. The best known achievable region for the general interference channel was defined by Han and Kobayashi [2]. The computation of the Han and Kobayashi formula for a general discrete memoryless channel is, in general, too complex. Recently, significant advances in obtaining upper bounds on the rate region have been made especially for the case of weak interference.

A $2\times2$ Gaussian interference channel in standard form (after suitable normalization) is given by

$$\mathbf{x} = \mathbf{H}\mathbf{s} + \mathbf{n}, \quad \mathbf{H} = \begin{bmatrix} 1 & \alpha_1 \\ \alpha_2 & 1 \end{bmatrix}, \tag{1}$$

where $\mathbf{s} = [s_1, s_2]^T$, and $\mathbf{x} = [x_1, x_2]^T$ are sampled values of the input and output signals, respectively. The noise vector $\mathbf{n}$ represents the additive Gaussian noises with zero mean and unit variance. The powers of the input signals are constrained to be less than $P_1, P_2$, respectively. The off-diagonal elements of $\mathbf{H}$, $\alpha_1, \alpha_2$ represent the degree of interference present. The major difference between the interference channel and the multiple access channel is that both encoding and decoding of each channel take place separately and independently, with no information sharing between receivers.

The capacity region of the Gaussian interference channel with very strong interference (i.e., $\alpha_1 \geq 1 + P_1$, $\alpha_2 \geq 1 + P_2$) is given by [23]

$$R_i \leq \log_2(1 + P_i), \; i = 1, 2. \tag{2}$$

This surprising result shows that very strong interference does not reduce users' capacity. A Gaussian interference channel is said to have strong interference if $\min\{\alpha_1, \alpha_2\} > 1$. Sato [24] derived an achievable capacity region (inner bound) of a Gaussian interference channel as the intersection of two multiple access Gaussian capacity regions embedded in the interference channel. The achievable region is the intersection of the rate pair of the rectangular region of the very strong interference (2) and the following region

> THE FACT THAT COMPETITIVE STRATEGIES CAN RESULT IN SIGNIFICANT DEGRADATION DUE TO THE PRISONER'S DILEMMA HAS BEEN CALLED THE PRICE OF ANARCHY.

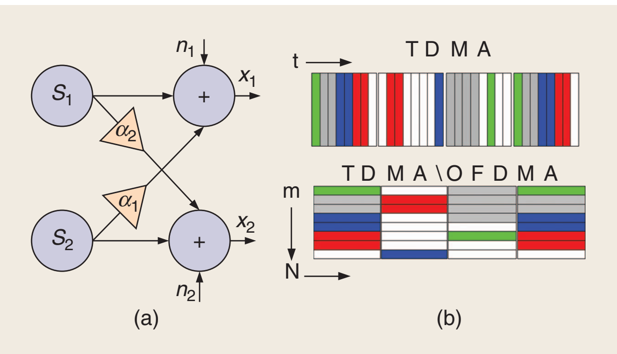
$$R_1 + R_2 \leq \log_2(\min\{1 + P_1 + \alpha_1 P_2, 1 + P_2 + \alpha_2 P_1\}). \quad (3)$$

While the two user flat interference channel is a well-studied (although unsolved) problem, much less is known about the frequency selective case. An $N \times N$ frequency selective Gaussian interference channel is given by

$$\mathbf{x}_k = \mathbf{H}_k \mathbf{s}_k + \mathbf{n}_k \quad k = 1, \ldots, K \quad (4)$$

where $\mathbf{H}_k = (h_{ij}(k))$ is the channel matrix at frequency $k$, $\mathbf{s}_k$, and $\mathbf{x}_k$ are sampled values of the input and output signal vectors at frequency $k$, respectively. The noise vector $\mathbf{n}_k$ represents an additive white Gaussian noise (AWGN) with zero mean and unit variance. The PSD of the input signals is constrained to be less than $p_1(k), p_2(k)$ respectively. Alternatively, only a total power constraint is given. The off-diagonal elements of $\mathbf{H}_k$ represent the degree of interference present at frequency $k$. The main difference between an interference channel and a multiple access channel (MAC) is that in the interference channel, each component of $s_k$ is decoded independently, and each receiver has access to a single element of $\mathbf{x}_k$. Therefore, iterative decoding schemes are much more limited and typically impractical for large numbers of users.

To overcome this problem, there are two simple strategies. When the interference is sufficiently weak, common wisdom is to treat the interference as noise, and code at a rate corresponding to the total noise. When the interference is stronger, i.e, the signal-to-interference ratio (SIR) is significantly lower than the signal-to-noise ratio (SNR), treating the interference as noise can be highly inefficient. One of the simplest ways to deal with medium-to strong-interference channels is through orthogonal signaling. Two extremely simple orthogonal schemes involve using FDM or TDM strategies. These techniques allow for single user detection (which will be assumed throughout this article) without the need for complicated multiuser detection. The loss in these techniques has been thoroughly studied, e.g., [23] in the constant channel case, showing degradation compared to the techniques requiring joint or sequential decoding. However, the widespread use of FDMA/TDMA as well as collision avoidance medium access control (carrier-sense multiple access) techniques make the analysis of these techniques very important from a practical point of view. For frequency selective channels, [also known as intersymbol interference (ISI) channels] both strategies can be combined by allowing time varying allocation of the frequency bands to different users as shown in Figure 1(b).




**[FIG1] (a) Standard form interference channel. (b) TDMA and joint TDMA/OFDMA.**

In this article, we limit ourselves to a joint FDM and TDM scheme where an assignment of disjoint portions of the frequency band to several transmitters is made at each time instance. This technique is widely used in practice because simple filtering can be used at the receivers' end to eliminate interference. All of these schemes operate under physical and regulation constraints such as the average power constraint and/or the PSD mask constraint.

## BASIC CONCEPTS IN COOPERATIVE AND COMPETITIVE GAME THEORY

In this section, we review the basic concepts of game theory in an abstract setting. Our focus is on concepts that have been found to be relevant to the frequency selective interference channel. We begin with competitive game theory and then continue to describe the cooperative solutions. The reader is referred to the excellent books by [25] and [26] for more details and for proofs of the main results mentioned here.

### *STATIC COMPETITIVE GAMES IN STRATEGIC FORM AND THE NASH EQUILIBRIUM*

A static $N$ player game in the strategic form is a three tuple $(N, A, \mathbf{u})$ composed of a set of players $\{1, \ldots, N\}$, a set of possible combinations of actions by each player denoted by $A = \prod_{n=1}^{N} A_n$, where $A_n$ is the set of actions for the $n$th player and a vector utility function $\mathbf{u} = [u_1, \ldots, u_N]$, where $u_n(a_1, \ldots, a_N): \prod_{n=1}^{N} A_n \rightarrow \mathcal{R}$ is the utility of the $n$th player when strategy vector $\mathbf{a} = (a_1, \ldots, a_N)$ has been played. The interpretation of $u_n$ is that player $n$ receives a payoff of $u_n(a_1, \ldots, a_N)$ when the players have chosen actions $a_1, \ldots, a_N$. The game is finite when for all $n$, $A_n$ is a finite set.

An important solution notion pertaining to games is the Nash equilibrium.

DEFINITION 1

A vector of actions $\mathbf{a} = (\mathrm{a}_1, \ldots, \mathrm{a}_N) \in A$ is a Nash equilibrium in pure strategies if and only if for each player $1 \leq n \leq N$ and

for every $\mathbf{a}'=(a_1',\ldots,a_N')$ such that $a_i'=\mathbf{a}_i$ for all $i\neq n$ and $a_n'\neq a_n$ we have $u_n(\mathbf{a}')<u_n(\mathbf{a})$, i.e., each player can only lose by deviating by himself from the equilibrium.

**CONVEX COMPETITIVE GAMES ARE A SPECIAL TYPE OF GAME THAT ARE CRUCIAL TO THE SPECTRUM MANAGEMENT PROBLEM.**

The Nash equilibrium in pure strategies does not always exist, as Example 1 shows.

### EXAMPLE 1: A GAME WITH NO PURE STRATEGY NASH EQUILIBRIUM

Consider the two-player game defined by the following: $A_i=\{0,1\}$. $u_i(a_1,a_2)=a_1\oplus a_2\oplus(i-1)$, i.e., the first player's payoff is one when actions are different and zero otherwise, while the second player's payoff is one when the actions are identical and zero otherwise. Clearly, this game, also known as matching pennies, has no Nash equilibrium in pure strategies, since one of the players can always improve his situation by changing his choice. Even when it exists, the Nash equilibrium in pure strategies is not necessarily unique, as Example 2 shows.

### EXAMPLE 2: A COMMUNICATION GAME WITH INFINITELY MANY PURE NE STRATEGIES

Assume that two users are sharing an AWGN multiple access channel (i.e., joint user decoding at the access point)

$$y=x_1+x_2+z, \tag{5}$$

where $z\sim N(0,\sigma^2)$ is a Gaussian noise. Each user has power $P$. It is well known [22] that the rate region of this multiple access channel is given by a pentagon defined by

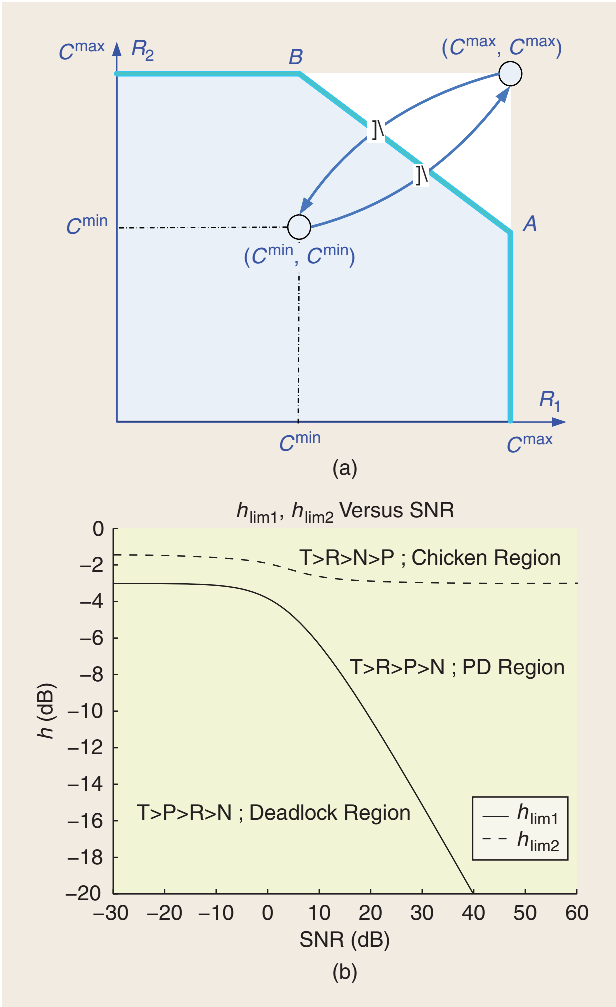
$$R_1\le C^{\max}\quad R_2\le C^{\max}\quad R_1+R_2\le C_{1,2}, \tag{6}$$

where $C^{\max}=1/2\log(1+P/\sigma^2)$ and $C_{1,2}=1/2\log(1+2P/\sigma^2)$. The corners $A, B$ [see Figure 2(a)] of the pentagon are $A=(C^{\max}, C^{\min})$ and $B=(C^{\min},C^{\max})$, where $C^{\min}=1/2\ \log\ (1+P/(P+\sigma^2))$, is the rate achievable by assuming that the other user's signal is interference. Note that any point on the line connecting the points $A, B$ is achieved by time sharing between these two points. Each player $n=1,2$ can choose a strategy $0\le\alpha_n\le 1$, which is the time sharing ratio between coding at this rate at point $A$ or $B$. The payoff in this game is given by

$$u_n(\alpha_1,\alpha_2)=\begin{cases}\alpha_n C^{\max}+(1-\alpha_n)C^{\min} & \text{if}\ \ \alpha_1+\alpha_2\le 1\\ 0 & \text{otherwise.}\end{cases} \tag{7}$$

The reason that the utility is 0 when $\alpha_1+\alpha_2>1$ is that no reliable communication is possible, since the rate pair achieved is outside the rate region. In this game, any valid strategy point such that $\alpha_1+\alpha_2=1$ is a Nash equilibrium. If user $n$ reduces his $\alpha_n$ obviously his rate is lower since he transmits a larger fraction of the time at the lower rate. If, on the other hand, he increases $\alpha_n$, then $\alpha_1+\alpha_2>1$ and both players achieve zero. Hence, the AWGN MAC game has an infinite number of Nash equilibrium points. A similar game was used in [27], where an infinite number of Nash equilibrium points are shown. It is interesting to note that a similar MAC game for the fading channel has a unique Nash equilibrium point [28].

To better understand this game, it is informative to look at the best response dynamics. The best response move is when a player attempts to maximize his utility against a given strategy vector. This is a well-established means of distributively achieving the Nash equilibrium. In the context of information theory, this strategy has been termed "iterative waterfilling" (IWF) [4]. If, in a multiple access game, the players use the best response simultaneously, the first step would be to transmit at $C^{\max}$. Each player then receives zero




**[FIG2] (a) Multiple access game with infinitely many pure strategy equilibria. Note that the parallel best response dynamics leads to unstable dynamics while the serial best response leads to one of two extremal NE points. (b) Graph of $h_{\lim1}$, $h_{\lim2}$. SNR, The solid line corresponds to $h_{\lim1}$ and the dashed line corresponds to $h_{\lim2}$ [9].**

utility and in the next step reduces his rate to $C^{\min}$, and vice versa. The iteration never converges and the utility of each player is given by $1/2C^{\min}$, worse than transmitting constantly at $C^{\min}$. Interestingly, in this case, the sequential best response leads to one of the points $A, B,$ which are the (non-axis) corners of the rate region. The moral of this story is that using the best response strategy should be done carefully even in multiple access scenarios such as in [29].

### *PURE AND MIXED STRATEGIES*

To overcome the first problem of the lack of equilibrium in pure strategies, the notion of mixed strategy has been proposed.

#### DEFINITION 2

A mixed strategy $\pi_n$ for player $n$ is a probability distribution over $A_n$.

The interpretation of mixed strategies is that player $n$ chooses his action randomly from $A_n$ according to the distribution $\pi_n$. The payoff of player $n$ in a game where mixed strategies $\pi_1, \ldots, \pi_N$ are played is the expected value of the utility

$$u_n(\pi_1, \ldots, \pi_N) = E_{\pi_1 \times \cdots \times \pi_N}[u_n(x_1, \ldots, x_N)]. \qquad (8)$$

#### EXAMPLE 3: MIXED STRATEGIES IN A RANDOM ACCESS GAME OVER A MULTIPLE ACCESS CHANNEL

To demonstrate the notion of mixed strategy, we now extend the multiple access game to a random multiple access game, where the players can choose with probability $p_n$ of working at rate $C^{\min}$ and $1 - p_n$ working at $C^{\max}$. This replaces the synchronized TDMA strategy in the previous game with a slotted random access protocol. This formulation allows for two pure strategies corresponding to the corner points $A, B$ and the mixed strategies amount to randomly choosing between these points. This game is a special case of the chicken dilemma (a termed proposed by B. Russell [30]), since for each user it is better to "chicken out" than to obtain zero rate when both players choose the tough strategies. Obviously, from the previous discussion, points $(C^{\max}, C^{\min})$ and $(C^{\min}, C^{\max})$ are in Nash equilibria. Simple computation shows that there is a unique Nash equilibrium in mixed strategies corresponding to $p_1 = p_2 = C^{\min}/C^{\max}$. Interestingly, the rates achieved by this random access (mixed strategy) approach are exactly $(C^{\min}, C^{\min})$, i.e., the price paid for random access is that both players achieve their minimal rate, so simple $p$-persistent random access provides no gain for the multiple access channel. Following Papadimitriou [8], we can call this the price of random access.

### *CONVEX GAMES*

Convex competitive games are a special type of game that are crucial to the spectrum management problem.

#### DEFINITION 3

An $N$ player game $(N, A, U)$ is convex if each $A_n$ is compact and convex and each $u_n(x_1, \ldots, x_n)$ is a convex function of $x_n$ for every choice of $\{x_j : j \neq n\}$.

Convex games always have a Nash equilibrium in pure strategies [31]. A simple proof can be found in [26]. Convex competitive games are especially important in the context of spectrum management, since the basic Gaussian interference game forms a convex game.

### *THE PRISONER'S DILEMMA*

The prisoner's dilemma game is the major problem in the competitive approach. It was first described by Flood and Dresher in 1950 almost immediately after the concept of Nash equilibrium was published [10]. For an overview of the prisoner's dilemma and its history, see Poundstone's excellent book [30]. It turns out that this game has a unique Nash equilibrium that is the stable point of the game. Moreover, this outcome is suboptimal for all players. The emergence of the prisoner's dilemma in simple symmetric interference channels was discussed in [9]. In [11] and [20], a characterization of cases where cooperative solutions are better for general interference channels are presented. We briefly describe a simple case where the prisoner's dilemma occurs [9]. We assume a simplified two-player game. The game is played over two frequency bands each with a symmetric interference channel. The channel matrices of this channel are $\mathbf{H}(1) = \mathbf{H}(2) = \mathbf{H}$ where

$$|h_{12}(1)|^2 = |h_{21}(1)|^2 = |h_{12}(2)|^2 = |h_{21}(2)|^2 = h$$

and $h_{ii}(k) = 1$. We limit the discussion to $0 \leq h < 1$. In our symmetric game, both users have the same power constraint $P$ and the power is allocated by $p_1(1) = (1-\alpha)P, p_1(2) = \alpha P$, $p_2(1) = \beta P, p_2(2) = (1-\beta)P$. We assume that the decoder treats the interference as noise and cannot decode it. The utility for user one given power allocation parameters $\alpha, \beta$ is given by its achievable rate

$$C^1 = \frac{1}{2}\log_2\left(1 + \frac{(1-\alpha)}{SNR^{-1} + \beta \cdot h}\right) + \frac{1}{2}\log_2\left(1 + \frac{\alpha}{SNR^{-1} + (1-\beta)\cdot h}\right) \qquad (9)$$

and similarly for user two, we replace $\alpha, \beta$. The set of strategies in this simplified game is $\{\alpha, \beta : 0 \leq \alpha, \beta \leq 1\}$. In the Nash equilibrium, each user allocates equal power to each band. It turns out that there are two functions, $h_{\lim_1}(\mathrm{SNR}), h_{\lim_2}(\mathrm{SNR})$ as described in Figure 2(b) and only three possible situations [9]:

1) Nash equilibrium is optimal for $h < h_{\lim_1}$
2) prisoner's dilemma, for $h_{\lim_1} < h < h_{\lim_2}$
3) a chicken dilemma, for $h_{\lim_2} < h$.

### *GENERALIZED NASH GAMES*

Games in strategic form are a very important part of game theory, and have many applications. However, in some cases the notion of a game does not capture all the complexities involved in the interaction between the players. Arrow and Debreu [32] defined the concept of a generalized Nash game

and a generalized Nash equilibrium. In strategic form games, each player has a set of strategies that is independent of the actions of the other players. However, in reality sometimes the actions of the players are constrained by the actions of the other players. The generalized Nash game or abstract economy concept captures this dependence.

**THE PRISONER'S DILEMMA GAME IS THE MAJOR PROBLEM IN THE COMPETITIVE APPROACH.**

### DEFINITION 4

A generalized Nash game with $N$ players is defined as follows: For each player $n$ we have a set of possible actions $X_n$ and a set function $K_n:\prod_{m\neq n} X_m \rightarrow P(X_n)$ where $P(X)$ is the power set of $X$, i.e, $K_n(\langle x_m : m \neq n\rangle) \subseteq X_n$ defines a subset of possible actions for player $n$ when other players play $\langle x_m : m \neq n\rangle$. $u_n(\mathbf{x})$ is a utility function defined on all tuples $(x_1, \ldots, x_N)$ such that $x_n \in K_n(x_m : m \neq n)$.

Similar to the definition of a Nash equilibrium, a generalized Nash equilibrium is explained in Definition 5.

### DEFINITION 5

A generalized Nash equilibrium is a point $\mathbf{x} = (x_1, \ldots, x_N)$ such that for all $n$ $x_n \in K_n(\langle x_m : m \neq n\rangle)$, and for all $\mathbf{y} = (y_1, \ldots, y_n)$ such that $y_n \in K_n(\langle x_m : m \neq n\rangle)$ and $y_m = x_m$ for $m \neq n$ $u_n(\mathbf{x}) \geq u_n(\mathbf{y})$.

Arrow and Debreu [32] proved the existence of a generalized Nash equilibrium under very limiting conditions. Their result was generalized, and currently the best result is by Ichiishi [33]. This result can be used to analyze certain fixed rate and margin versions of the iterative waterfilling algorithm [6] as will be shown in the next section.

## *NASH BARGAINING THEORY*

The prisoner's dilemma highlights the drawbacks of competition due to the players' mutual mistrust. It is apparent that the essential condition for cooperation is that it should generate a surplus, i.e., an extra gain that can be divided between the parties. Bargaining is essentially the process of distributing the surplus. Thus, bargaining is first and foremost a process in which parties seek to reconcile contradictory interests and values. If all players commit to following the rules, the key question is then "what reasonable outcome will be acceptable to all parties?" Therefore, the players need to agree on a bargaining mechanism that they must stick with during the negotiations. The bargaining results may be affected by factors such as the power of each party, the amount of information available to each of the players, and the delay response of the players. Nash [10], [34] introduced an axiomatic approach based on properties that a valid outcome of the bargaining should satisfy. This approach proved to be very useful since it succeeded in finding a unique solution through a small number of simple conditions (axioms), thus obviating the need for the complicated bargaining process once all the parties had accepted these conditions.

We now define the bargaining problem. An $n$-player bargaining problem is described as a pair $\langle S, d\rangle$, where $S$ is a convex set in $n$-dimensional Euclidean space, consisting of all feasible sets of $n$-player utilities that the players can receive when cooperating. The disagreement point $d$ is an element of $S$, representing the outcome if the players fail to reach an agreement. Furthermore, $d$ can also be viewed as the utilities resulting from a noncooperative strategy (competition) between players, which is the Nash equilibrium of a competitive game. The assumption that $S$ is a convex set is reasonable if both players select cooperative strategies, since the players can use alternating or mixed strategies to achieve convex combinations of pure bargaining outcomes. Given a bargaining problem, we say that the vector $\mathbf{u} \in S$ is Pareto optimal if for all $\mathbf{w} \in S$ there is no $\mathbf{u} \in S$ such that $\mathbf{w} \leq \mathbf{u}$ (coordinate-wise). A solution to the bargaining problem is a function $F$ defined on all bargaining problems such that $F(\langle S, d\rangle) \in S$. Nash's axiomatic approach is based on the following four axioms which the solution function $F$ should satisfy the following:

- *Linearity (LIN)*: Assume that we consider the bargaining problem $\langle S', d'\rangle$ obtained from the problem $\langle S, d\rangle$ by transformations: $s_n' = \alpha_n s_n + \beta_n$, $n = 1, \ldots, N$. $d_n' = \alpha_n d_n + \beta_n$. Then the solution satisfies $F_i(\langle S', d'\rangle) = \alpha_n F_n(\langle S, d\rangle) + \beta_n$, for all $n = 1, \ldots, N$.
- *Symmetry (SYM)*: If two players $m < n$ are identical in the sense that $S$ is symmetric with respect to changing the $m$th and the $n$th coordinates, then $F_m(\langle S, d\rangle) = F_n(\langle S, d\rangle)$. Equivalently, players who have identical bargaining preferences get the same outcome at the end of the bargaining process.
- *Pareto optimality (PAR)*: If $\mathbf{s}$ is the outcome of the bargaining, then no other state $\mathbf{t}$ exists such that $\mathbf{s} < \mathbf{t}$ (coordinate wise).
- *Independence of irrelevant alternatives (IIA)*: If $S \subseteq T$ and if $F(\langle T, d\rangle) = (u_1^*, u_2^*) \in S$, then $F(\langle S, d\rangle) = (u_1^*, u_2^*)$.

Surprisingly these simple axioms fully characterize a bargaining solution known as Nash's bargaining solution (NBS).

Based on these axioms and definitions we now can state Nash's theorem [10].

### THEOREM 1

Assume that $S$ is compact and convex, then there is a unique bargaining solution $F(\langle S, d\rangle)$, satisfying the axioms INV, SYM, IIA, and PAR, which is given by

$$F(\langle S, d\rangle) = \arg \max_{(d_1,\ldots,d_N)\leq(s_1,\ldots,s_N)\in S} \prod_{n=1}^{N}(s_n - d_n). \qquad (10)$$

Before continuing our examination of the bargaining solution, we add a cautionary note. Although Nash's axioms are mathematically appealing, they may not be acceptable in some scenarios. Indeed, various alternatives to these axioms have

been proposed that lead to other solution concepts. An extensive survey of these solutions can be found in [35]. In the communication context, the axioms proposed by Boche et al. [12] lead to a generalized NBS. To demonstrate the application of the NBS to interference channels, we begin with a simple example for flat channels.

**THE BARGAINING RESULTS MAY BE AFFECTED BY FACTORS SUCH AS THE POWER OF EACH PARTY, THE AMOUNT OF INFORMATION AVAILABLE TO EACH OF THE PLAYERS, AND THE DELAY RESPONSE OF THE PLAYERS.**

### EXAMPLE 3

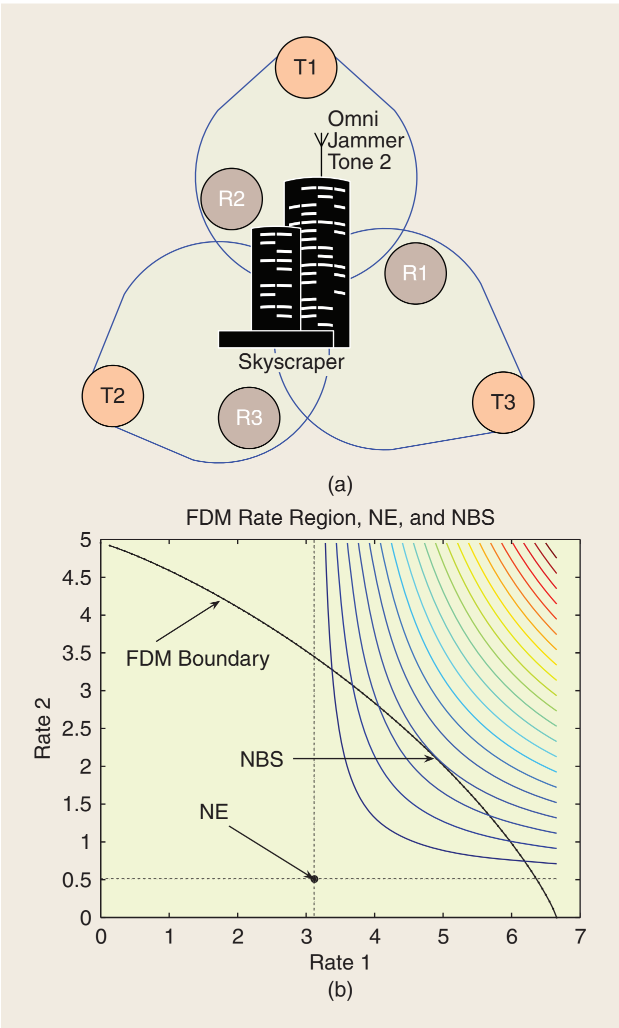
Consider two players communicating over a 2×2 memoryless Gaussian interference channel with bandwidth $W = 1$, as described in (1). Assume for simplicity that $P_1 = P_2 = P$. We assume that no receiver can perform joint decoding, and the utility of player $n$, $U_n$, is given by the achievable rate $R_n$. Similar to the prisoner's dilemma example, if the players choose to compete, the competitive strategies in the interference game are given by a flat power allocation and the resulting rates are given by $R_{nC} = 1/2 \log_2(1 + P/(1+\alpha_n^2 P))$. Since the rates $R_{nC}$ are achieved by competitive strategy, player $n$ will cooperate only if he obtains a rate higher than $R_{nC}$. The game theoretic rate region is defined by pair rates higher than the competitive rates $R_{nC}$ [see Figure 3(b)]. Assume that the players agree on using only FDM cooperative strategies, i.e., player $n$ uses a fraction of $0 \le \rho_n \le 1$ of the band. If we consider only Pareto optimal strategy vectors, then obviously $\rho_2 = 1 - \rho_1$. The rates obtained by the two users are given by $R_n(\rho_n) = \rho_n/2 \log_2(1 + P/\rho_n)$. The two users will benefit from the FDM type of cooperation as long as

$$R_{nC} \le R_n(\rho_1), \quad n = 1, 2. \tag{11}$$

The FDM NBS is given by solving the problem

$$\max_{\rho} F(\rho) = \max_{\rho} (R_1(\rho) - R_{1C})(R_2(\rho) - R_{2C}). \tag{12}$$

The cooperative solution for this flat channel model was derived in [11] and [20].




**[FIG3] (a) A frequency selective channel with a unique NE, where both sequential and parallel IWF diverge. (b) The game theoretic rate region is defined by the set of rates that are better than the competitive equilibrium. The boundary of the region is exactly the set of Pareto-optimal points. The hyperbola tangent to the rate region defines the NBS.**

## APPLICATION OF GAME THEORY TO FREQUENCY SELECTIVE INTERFERENCE CHANNELS

In this section, we apply the ideas presented in previous sections to analyze the frequency selective interference game. We provide examples for both competitive and cooperative game theoretic concepts.

### *WATERFILLING-BASED SOLUTIONS AND THE NASH EQUILIBRIUM*

To analyze the competitive approach to frequency selective interference channels, we first define the discrete-frequency Gaussian interference game, which is a discrete version of the game defined in [4]. Let $f_0 < \cdots < f_K$ be an increasing sequence of frequencies. Let $I_k$ be the closed interval given by $I_k = [f_{k-1}, f_k]$. We now define the approximate Gaussian interference game denoted by $GI_{\{I_1,\ldots,I_K\}}$.

Let players $1, \ldots, N$ operate over $K$ parallel channels. Assume that the $K$ channels have coupling functions $h_{ij}(k)$. Assume that user $i$ is allowed to transmit a total power of $P_i$. Each player can transmit a power vector $\mathbf{p}_n = (p_n(1), \ldots, p_n(K)) \in [0, P_n]^K$ such that $p_n(k)$ is the power transmitted in interval $I_k$. Therefore we have $\sum_{k=1}^{K} p_n(k) = P_n$. The equality follows from the fact that in a non-cooperative scenario all users will use all the available power. This implies that the set of power distributions for all users is a closed convex subset of

the hypercube $\prod_{n=1}^{N}[0, P_n]^K$ given by $B = \prod_{n=1}^{N} B_n$ where $B_n$ is the set of admissible power distributions for player $n$. Each player chooses a PSD $\mathbf{p}_n = \langle p_n(k) : 1 \le k \le N \rangle \in B_n$. Let the payoff for user $i$ be given by the capacity available to player $n$

$$C_n(\mathbf{p}_1, \ldots, \mathbf{p}_N) = \sum_{k=1}^{K} \log_2(1 + \mathrm{SINR}(k)), \tag{13}$$

where

$$\mathrm{SINR}(k) = \frac{|h_{nn}(k)|^2 p_n(k)}{\sum |h_{nm}(k)|^2 p_m(k) + \sigma_n^2(k)}. \tag{14}$$

is the signal to interference ratio for user $n$ at frequency $k$, $\mathbf{p}_1, \ldots, \mathbf{p}_N$ are the power distributions, $h_{nm}(f)$ are the channel and crosstalk responses and $\sigma_n^2(k) > 0$ is external noise present at the $i$th channel receiver at frequency $k$. In cases where $\sigma_n^2(k) = 0$, capacities might become infinite using FDM strategies; however, this is a nonphysical situation due to receiver noise that is always present, even if it is small. Each $C_n$ is continuous in all variables.

### DEFINITION 6
The Gaussian interference game $GI_{\{I_1, \ldots, I_k\}} = \{N, \mathbf{B}, \mathbf{C}\}$ is the $N$ $N$-players noncooperative game with payoff vector $\mathbf{C} = (C_1, \ldots, C_N)$ where $C_n$ is defined in (13) and $\mathbf{B}$ is the strategy set.

The interference game is a convex noncooperative $N$-person game, since each $\mathbf{B}_i$ is compact and convex and each $C_n(\mathbf{p}_1, \ldots, \mathbf{p}_N)$ is continuous and convex in $\mathbf{p}_n$ for any value of $\{\mathbf{p}_m, m \neq n\}$. Therefore, it always has a Nash equilibrium in pure strategies. A presentation of the proof in this case using a waterfilling interpretation is given in [36].

A much harder problem is the uniqueness of Nash equilibrium points in the waterfilling game. This is very important to the stability of waterfilling strategies. An initial result was reported in [37]. A more general analysis of the convergence was given in [4], [5], and [38]–[40]. Even the uniqueness of the Nash equilibrium does not imply a stable dynamics. Scutari et al. [5] provided conditions for convergence. Basically, they use the Banach fixed-point theorem, and require that the waterfilling response be a contractive mapping. The waterfilling process has several versions: the sequential IWF is performed by a single player at each step. The parallel IWF is performed simultaneously by all players at each step, and the asynchronous IWF is performed in an arbitrary order. For a detailed discussion of the convergence of these techniques, see [5]. It should be emphasized that some conditions on the interference channel matrices are indeed required. A simple condition is strong diagonal dominance [4], but other papers have relaxed these assumptions. In all typical DSL scenarios, the IWF algorithms converge. However, the convergence conditions are not always met, even in very simple cases, as the following examples show.

### EXAMPLE 4: DIVERGENCE OF THE PARALLEL IWF
We consider a Gaussian interference game with two tones and three players. Each player has total power $P$. The signal received by each receiver is just $y_n(k) = \sum_{m=1}^{3} x_m(k) + z_n(k)$ where $z_n(k)\, N(0, \sigma^2(k))$, where the noise in the second band is stronger satisfying $\sigma^2(2) = P + \sigma^2(1)$. We assume that simultaneous waterfilling is used. In the first stage, all users put all their power into frequency one, by the first inequality. In the second stage, all users see a noise and interference power of $2P + \sigma^2(1)$ at the first frequency, while the interference at the second frequency is $\sigma^2(2) = P + \sigma^2(1)$. Since even when all power is put into frequency two, the total power plus noise is below the noise level at frequency one and all users will move their total power to frequency two. This will continue, with all users alternating between frequencies simultaneously. The average rate obtained by the simultaneous iterative waterfilling is

$$\frac{1}{4}\log\left(1 + \frac{P}{2P + \sigma^2(1)}\right) + \frac{1}{4}\log\left(1 + \frac{P}{3P + \sigma^2(1)}\right).$$

A Nash equilibrium exists in this case, for example, if two users use frequency one and one user uses frequency two, resulting in a rate

$$\frac{1}{2}\log\left(1 + \frac{P}{P + \sigma^2(1)}\right)$$

for each user.

The previous example demonstrated a simple condition where one of the waterfilling schemes diverges. The following channel is frequency selective, with a single Nash equilibrium in the Gaussian interference game. However none of the water filling schemes converge.

### EXAMPLE 5: DIVERGENCE OF ALL WATERFILLING APPROACHES
We now provide a second example where both the sequential and the parallel IWF diverge, even though there is a unique Nash equilibrium point. Assume that we have two channels where the channel matrices $\mathbf{H}(k)$, $k = 0, 1$ are equal and given by

$$\mathbf{H}(k) = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \tag{15}$$

and the noise at the first tone is $\sigma^2$ and at the second tone is $\sigma^2 + P$. Each user has total power $P$. This situation might occur when there is a strong interferer at tone two while the receivers and transmitters are located on the sides of a triangle, with each user transmitting to a receiver near the next transmitter as in Figure 3(a). When the first user allocates his power, he puts it all at the first frequency. The second user chooses tone two. The third player puts all his power at frequency one but this generates interference to user one who migrates to frequency two with the outcome that the users change their transmission tone at each step. In the simultaneous IWF, all users choose channel one and then migrate together to channel two and back. It is worth noting that this game has a unique equilibrium, where each user allocates

two-thirds of the power to frequency one and one-third of the power to frequency two. Nevertheless all iterative schemes diverge.

### PRICING MECHANISMS FOR REGULATING DISTRIBUTED SOLUTIONS

To overcome some of the problems of competitive behavior, regulation can play an important role. One generalization of the RA-IWF algorithm is the band preference spectrum management (BPSM) algorithm [41] in which each user solves the following problem in parallel (or sequentially) to the other users:

$$\begin{aligned} &\max_{p_n(1),\ldots,p_n(K)} && \textstyle\sum_{k=1}^{K} c_n(k)\log_2(1+\mathrm{SINR}(k)) \\ &\text{Subject to} && P_n = \textstyle\sum_{k=1}^{K} p_n(k)\,. \end{aligned} \tag{16}$$

In the BPSM algorithm, the total rate is replaced by a weighted sum of the rates at each frequency. The weights can be provided by the regulator to limit the use of certain bands by strong users, so that users that suffer severe interference but do not affect other users are protected. This results in waterfilling against a compensated noise level.

An alternative approach to the BPSM is using generalized Nash games. The basic approach was proposed in [4] and termed "fixed-margin" (FM) IWF. Each user has a power constraint, a target noise margin, and a desired rate. The user minimizes his power as long as he achieves his target rate. This is a generalized Nash game, first analyzed by Pang et al. [6], who provided the first conditions for convergence. Noam and Leshem [7] proposed a generalization of the FM-IWF termed "iterative power pricing" (IPP). In their solution, a weighted power is minimized, where frequency bands that have higher capacity are reserved for players with longer lines through a line dependent pricing mechanism. The users iteratively optimize their power allocation so that their rates and total power constraints are maintained while minimizing the total weighted power. It can be shown that the conditions in Pang et al. can also be used to analyze the IPP algorithm. For both the BPSM and the IPP approaches, simple pricing schemes that are adapted to the DSL scenario have been proposed. The general question of finding good pricing schemes is still open but would require combining physical modeling of the channels as well as insights into the utilities as a function of the desired rate. Even the autonomous spectrum balancing (ASB) algorithm [42] can be viewed as a generalized Nash game, where the utility is given by the rate of a fictitious reference line, and the strategy sets should satisfy both the rate and power constraint. The drawback of the ASB approach here is finding a reference line which serves as a good utility function.

### BARGAINING OVER FREQUENCY SELECTIVE CHANNELS UNDER A MASK CONSTRAINT

In this section, we define the cooperative game corresponding to the joint FDM/TDM achievable rate region for the frequency selective $N$-user interference channel. For simplicity of presentation, we limit the derivation to the two-player case under the PSD mask constraint. In [18], the NBS was used for resource allocation in orthogonal frequency division multiple access (OFDMA) systems. The goal was to maximize the overall system rate, under constraints on each user's minimal rate requirement and total transmitted power. However, in that paper, the solution was used only as a measure of fairness. Therefore, the point of disagreement was not taken as the Nash equilibrium for the competitive game, and instead an arbitrary $R_{\min}$ was used. This can lead to a nonfeasible problem, and the proposed algorithm might become unstable. An alternative approach is based on the PSD mask constraint [20] in conjunction with the general bargaining theory originally developed by Nash [10], [34]. Based on the solution for the PSD limited case, computing the NBS under total power constraint can then be solved efficiently as well [21]. To keep the discussion simple, we concentrate on the two user PSD mask limited case.

In real applications, the regulator limits the PSD mask and not only the total power constraint. Let the $K$ channel matrices at frequencies $k = 1, \ldots, K$ be given by $\langle \mathrm{H}_k : k = 1, \ldots, K\rangle$. A player is allowed to transmit at maximum power $p_n(k)$ in the $k$th frequency bin. In the competitive scenario, under the mask constraint, all players transmit at the maximal power they can use. Thus, player $n$ chooses the PSD, $\mathbf{p}_n = \langle p_n(k) : 1 \le k \le K\rangle$. The payoff for user $n$ in the noncooperative game is therefore given by:

$$R_{nC}(\mathbf{p}) = \sum_{k=1}^{K} \log_2(1 + \mathrm{SINR}(k)). \tag{17}$$

Here, $R_{nC}$ is the capacity available to player $n$ given a PSD mask constraint distribution $\mathbf{p}$. Note that without loss of generality, and to simplify notation, we assume that the width of each bin is normalized to one. We now define the cooperative game $G_{TF}(2, K, \mathbf{p})$.

#### DEFINITION 7

The FDM/TDM game $G_{TF}(2, K, \mathbf{p})$ is a game between two players transmitting over $K$ frequency bins under the common PSD mask constraint. Each player has full knowledge of the channel matrices $\mathrm{H}_k$ and the following conditions hold:

1) Player $n$ transmits using a PSD limited by $\langle p_n(k) : k = 1, \ldots, K\rangle$.
2) The players use coordinated FDM/TDM strategies. The strategy for player one is a vector $\boldsymbol{\alpha}_1 = [\alpha_1(1), \ldots, \alpha_1(K)]^T$ where $\alpha_1(k)$ is the proportion of time player one uses the $k$th frequency channel. Similarly, the strategy for player two is a vector $\boldsymbol{\alpha}_2 = [\alpha_2(1), \ldots, \alpha_2(K)]^T$.
3) The utility of the players is given by

$$R_n(\boldsymbol{\alpha}_n) = \sum_{k=1}^{K} \alpha_n(k) R_n(k),$$

where

$$R_n(k) = \log_2\left(1 + \frac{|h_{nn}(k)|^2 p_n(k)}{\sigma_n^2(k)}\right). \tag{18}$$

By Pareto optimality of the NBS, for each $k$, $\alpha_2(k) = 1 - \alpha_1(k)$, so we will only refer to $\boldsymbol{\alpha} = \boldsymbol{\alpha}_1$ as the strategy for both players. Note that interference is avoided by time sharing at each frequency band; i.e., only one player transmits with maximal power at a given frequency bin at any time. The allocation of the spectrum using the vector $\boldsymbol{\alpha}$ induces a simple convex optimization problem that can be posed as follows:

$$\begin{aligned} &\max\,(R_1(\boldsymbol{\alpha}) - R_{1C})(R_2(\boldsymbol{\alpha}) - R_{2C}) \\ &\text{subject to: } 0 \leq \alpha(k) \leq 1 \ \ \forall k, R_{nC} \leq R_n(\boldsymbol{\alpha}) \ \ \forall n. \end{aligned} \tag{19}$$

Since the log of the Nash function (19) is a convex function the overall problem is convex. Hence, it can be solved efficiently using Karush-Kuhn-Tucker (KKT) conditions [20]. Assuming that a feasible solution exists it follows from the KKT conditions that the allocation is done according to the following rules:

1) The two players share frequency bin $k$, $(0 < \alpha(k) < 1)$ if

$$\frac{R_1(k)}{R_1(\boldsymbol{\alpha}) - R_{1C}} = \frac{R_2(k)}{R_2(\boldsymbol{\alpha}) - R_{2C}}. \tag{20}$$

2) Only player $n$ is using the frequency bin $k$, $(\alpha_n(k) = 1)$, if

$$\frac{R_n(k)}{R_n(\boldsymbol{\alpha}) - R_{nC}} > \frac{R_{3-n}(k)}{R_{3-n}(\boldsymbol{\alpha}) - R_{3-n,C}}. \tag{21}$$

These rules can be further simplified. Let $L_k = R_1(k)/R_2(k)$ be the ratio between the rates at each frequency bin. We can sort the frequency bins in decreasing order according to $L_k$. From now on we assume that when $k_1 < k_2$ then $L_{k_1} > L_{k_2}$. If all the values of $L_k$ are distinct, then there is at most a single frequency bin that has to be shared between the two players. Since only one bin can satisfy equation (20), let us denote this frequency bin as $k_s$, then all the frequency bins $1 \leq k < k_s$ will only be used by player one, while all the frequency bins $k_s < k \leq K$ will be used by player two. The frequency bin $k_s$ must be shared according to the rules.

We now need to find the frequency bin that must be shared between the players if there is a solution. Let us define the surplus of players one and two when using the NBS as $A = \sum_{m=1}^{K} \alpha(m) R_1(m) - R_{1C}$, and $B = \sum_{m=1}^{K}(1 - \alpha(m)) R_2(m) - R_{2C}$, respectively. Note that the ratio $\Gamma = A/B$ is independent of frequency and is set by the optimal assignment. A-priori $\Gamma$ is unknown and may not exist. We are now ready to define the optimal assignment of the $\alpha$s.

Let $\Gamma_k$ be a moving threshold defined by $\Gamma_k = A_k/B_k$, where

$$A_k = \sum_{m=1}^{k} R_1(m) - R_{1C}, B_k = \sum_{m=k+1}^{K} R_2(m) - R_{2C}. \tag{22}$$

$A_k$ is a monotonically increasing sequence, while $B_k$ is monotonically decreasing. Hence, $\Gamma_k$ is also monotonically increasing. $A_k$ is the surplus of player one when frequencies $1, \ldots, k$ are allocated to player one. Similarly $B_k$ is the surplus of player two when frequencies $k+1, \ldots, K$ are allocated to player two.

Let $k_{\min} = \min_k\{k : A_k \geq 0\}$, and $k_{\max} = \min_k\{k : B_k < 0\}$. Since we are interested in feasible NBS, we must have a positive surplus for both users. Therefore, using the allocation rules, we obtain $k_{\min} \leq k_{\max}$ and $L_{k_{\min}} \leq \Gamma \leq L_{k_{\max}}$. The sequence $\{\Gamma_m : k_{\min} \leq m \leq k_{\max} - 1\}$ is strictly increasing, and always positive. While the threshold $\Gamma$ is unknown, one can use the sequences $\Gamma_k$ and $L(k)$ to find the correct $\Gamma$.

If there is an NBS, let $k_s$ be the frequency bin that is shared by the players. Then, $k_{\min} \leq k_s \leq k_{\max}$. Since both players must have a positive gain in the game ( $A > A_{k_{\min}-1}, B > B_{k_{\max}}$ ). Let $k_s$ be the smallest integer such that $L(k_s) < \Gamma_{k_s}$, if such $k_s$ exists. Otherwise, let $k_s = k_{\max}$.

**[TABLE 1] ALGORITHM FOR COMPUTING THE 2 × 2 FREQUENCY SELECTIVE NBS.**

```
INITIALIZATION: SORT THE RATIOS L(k) IN DECREASING ORDER.
CALCULATE THE VALUES OF A_k, B_k AND Γ_k, k_min, k_max,

IF k_min > k_max NO NBS EXISTS. USE COMPETITIVE SOLUTION.
ELSE
  FOR k = k_min TO k_max − 1
    IF L(k) ≤ Γ_k.
      SET k_s = k AND α'S ACCORDING TO THE LEMMAS-THIS IS NBS. STOP
    END
  END
  IF NO SUCH k EXISTS, SET k_s = k_max AND CALCULATE g.
  IF g ≥ 0 SET α_{k_s} = g, α(k) = 1, FOR k < k_max. STOP.
  ELSE (g < 0)
  THERE IS NO NBS. USE COMPETITIVE SOLUTION.
  END.
END
```

## LEMMA 1

The following two statements provide the solution:

1) If an NBS exists for $k_{\min} \leq k_s < k_{\max}$, then $\alpha(k_s)$ is given by $\alpha(k_s) = \max\{0, g\}$, where

$$g = 1 + \frac{B_{k_s}}{2R_2(k_s)}\left(1 - \frac{\Gamma_{k_s}}{L(k_s)}\right). \tag{23}$$

2) If an NBS exists and there is no such $k_s$, then $k_s = k_{\max}$ and $\alpha(k_s) = g$.

Based on the previous lemmas, the algorithm is described in Table 1. In the first stage, the algorithm computes $L(k)$ and sorts them in a nonincreasing order. Then $k_{\min}$, $k_{\max}$, $A_k$, and $B_k$ are computed. In the second stage, the algorithm computes $k_s$ and $\boldsymbol{\alpha}$.

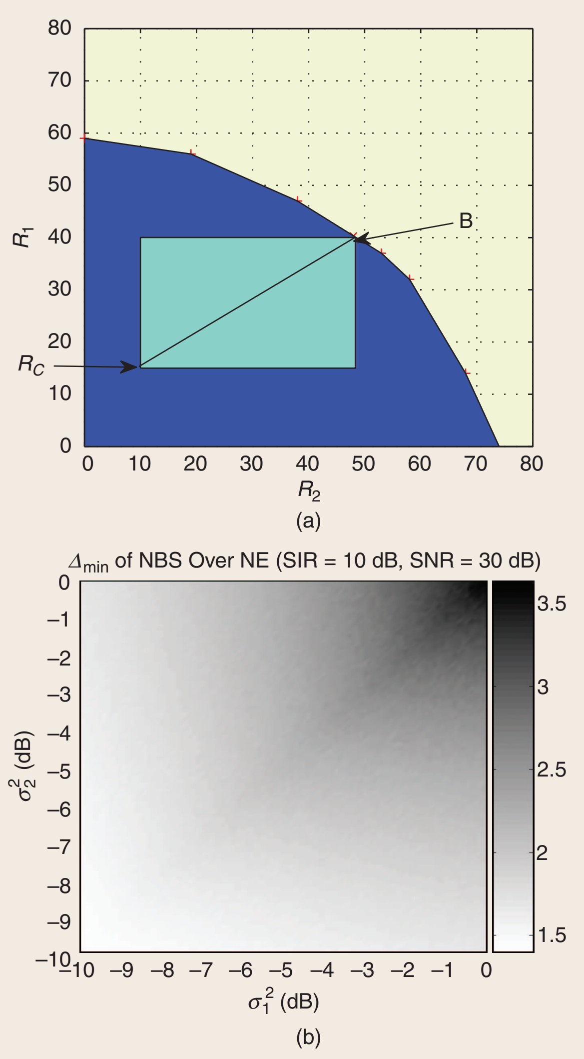
To demonstrate the algorithm we compute the Nash bargaining for Example 6.

## EXAMPLE 6

Consider two players communicating over a 2×2 Gaussian interference channel with six frequency bins. The players' rates (if they do not cooperate) are $R_{1C} = 15$, and $R_{2C} = 10$. The feasible rates $R_1(k)$ and $R_2(k)$ in each frequency bin with no

**[TABLE 2] THE RATES OF THE PLAYERS IN EACH FREQUENCY BIN AFTER SORTING.**

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $R_1$ | 14 | 18 | 5 | 10 | 9 | 3 |
| $R_2$ | 6 | 10 | 5 | 15 | 19 | 19 |
| $L(k)$ | 2.33 | 1.80 | 1.00 | 0.67 | 0.47 | 0.16 |
| $A_k$ | −1 | 17 | 22 | 32 | 41 | 44 |
| $B_k$ | 58 | 48 | 43 | 28 | 9 | −10 |
| $\Gamma_k$ | −0.02 | 0.35 | 0.51 | 1.14 | 4.56 | −4.40 |




**[FIG4] (a) Feasible FDM rate region (blue area), NBS (the area covered by the light blue rectangle). (b) Per user price of anarchy for frequency selective Rayleigh fading channel. SNR = 30 dB.**

interference are given in Table 2 after sorting the frequency bins with respect to $L_k$.

We now need to compute the surplus $A_k$ and $B_k$ for each player. If the NBS exists, then the players must have a positive surplus; thus, $k_{\min} = 2$ and $k_{\max} = 4$. Since $k = 4$ is the first bin such that $\Gamma_k > L_k$, we can conclude that $k_s = 4$ and $\alpha = 0.33$ (using lemma 4.1). Thus, player one is using frequency bins 1, 2, and 3, and using one-third of the time, frequency bin 4.

We can also give a geometric interpretation to the solution. In Figure 4(a), we draw the feasible total rate that player one can obtain as a function of the total rate of player two. The enclosed area in blue is the achievable utilities set. Since the frequency bins are sorted according to $L_k$, the set is convex. Point $R_C = (R_{2C}, R_{1C}) = (10, 15)$ is the point of disagreement. If point $R_C$ is inside the achievable utility set there is a solution. The slope of the boundaries of the achievable utilities set with respect to the $-x$ axis is $L_k$. The vector $R_C - B$ connects point $R_C$ and point B with the same slope with respect to the $x$ axis; this is the geometric interpretation of (20). The area covered by the light blue rectangle is the value of the Nash product function.

The results can be generalized in the following directions.

First, if the values $\{L(k) : k = 1, \ldots, K\}$ are not all distinct then if there is a solution one can always find an allocation such that at most a single frequency has to be shared.

Second, in the general case of $N$ players the optimization problem has similar KKT conditions and can be solved using a convex optimization algorithm. Moreover, the optimal solution has at most $\binom{N}{2}$ frequencies that are shared between different players. This suggests that the optimal FDM NBS is very close to the joint FDM/TDM solution. It is obtained by allocating the common frequencies to one of the users. Third, while the method described above fits stationary channels well, the method is also useful when only fading statistics are known. In this case, the coding strategy will change, and the achievable rate in the competitive case and the cooperative case are given by

$$\tilde{R}_{nC}(\mathbf{p}_i) = \sum_{k=1}^{K} E\left[\log_2\left(1 + \frac{|h_{nn}(k)|^2 p_n(k)}{\sum_{m\neq n}|h_{nm}(k)|^2 p_m(k) + \sigma_n^2(k)}\right)\right]$$

$$\tilde{R}_n(\boldsymbol{\alpha}_n) = \sum_{k=1}^{K} \alpha_n(k) E\left[\log_2\left(1 + \frac{|h_{nn}(k)|^2 \rho_n(k)}{\sigma_n^2(k)}\right)\right], \tag{24}$$

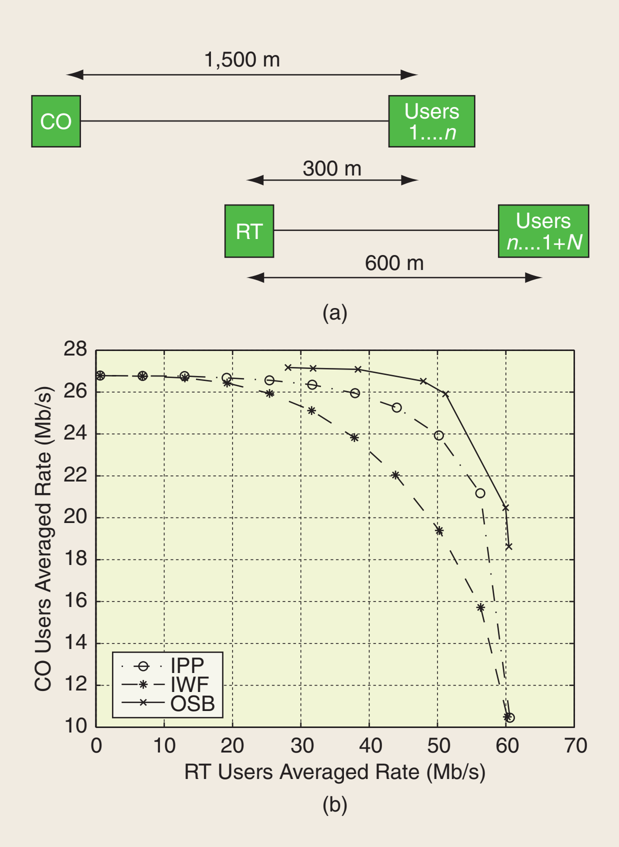
respectively. All the rest of the expansion is unchanged, replacing $R_{nC}$ and $R_n(\boldsymbol{\alpha}_n)$ by $\tilde{R}_{nC}$, $\tilde{R}_i(\boldsymbol{\alpha}_n)$, respectively. This is particularly attractive when the computations are done in a distributed fashion. In this case, only channel state distributions are sent between the units. Hence, the time scale for this data exchange is much longer. This implies that the method can be used without a central control, by exchange of parameters between the units at a very low rate.

Fourth, computing the NBS under total power constraint is more difficult to solve. Several ad-hoc techniques have been proposed in the literature. Recently, it was shown that for this case there is an algorithm that can find the optimal solution [21].

## APPLICATIONS

### *WEAK INTERFERENCE: THE DSL CASE*

The DSL channel is an interesting example for testing algorithms emerging from game theoretic considerations. The iterative waterfilling algorithm [4] has been successfully implemented for distributed spectrum coordination of DSL lines. However, the drawbacks caused by the prisoner's dilemma suggest that the strictly competitive approach (RA-IWF) is inappropriate for real-life applications. Several amendments have been proposed. The first is FM-IWF [4]. In this algorithm, the players are provided with a fixed target rate and each user independently minimizes his total transmit power. As shown by Pang et al. [6], this is a generalized Nash game that converges if the interference is sufficiently weak. In [7], a generalization of the FM-IWF is proposed that favors weak users who implement a pricing mechanism termed "iterative power pricing." This pricing mechanism improves the performance of the FM-IWF. The game theoretic approaches exhibit very good performance as compared to optimal spectrum management techniques, as shown in Figure 5.




**[FIG5] (a) Simulation setup. Three users at each location. (b) Rate regions of FM IWF, IPP, and OSB. The optimal OSB is centralized and computationally very expensive. It is given as a reference for the optimal performance.**

### *MEDIUM AND STRONG INTERFERENCE–WIRELESS TECHNOLOGIES*

The rapid adoption of wireless services by the public has resulted in a remarkable increase in demand for reliable high data rate Internet access. This process has motivated the development of new technologies. The new generation of cellular systems like LTE and WiMax operating in the licensed band will be launched in the next five years. In the unlicensed band, 802.11N with MIMO technology will soon become part of our daily lives. The capacity of future wireless data networks will inevitably be interference-limited due to the limited radio spectrum. It is clear that any cooperation between the different networks or base stations sharing the same spectral resource will be a source of significant improvement in the utilization of radio resources. Even in the same cell, cooperation between sectors can improve the overall spectral efficiency (b/Hz/sec./sector). The OFDMA technology is capable of efficiently allocating frequency bins based on the channel response of the user. In [43], a noncooperative game approach was employed for distributed subchannel assignment, adaptive modulation, and power control for multicell OFDM networks. The goal was to minimize the overall transmitted power under maximal power and per user minimal rate constraints. Based on simulation results, the proposed distributed algorithm reduces the overall transmitted power in comparison to a pure waterfilling scheme for a seven-cell case. Kwon and Lee [44] presented a distributed resource allocation algorithm for multicell OFDMA systems relying on a noncooperative game in which each base station tries to maximize the system performance while minimizing the cochannel interference. They proved that there exists a Nash equilibrium point for the noncooperative game and the equilibrium is unique in some constrained environment. However, the Nash equilibrium achieved by the distributed algorithm may not be as efficient as the resource allocation obtained through centralized optimization. To demonstrate the advantage of the NBS over competitive approaches for a frequency selective interference channel, we assumed that two users share a frequency selective Rayleigh fading channel. The direct channels have a unit fading variance and an SNR of 30 dB. The users suffer from cross interference. The cross channels fading variance was varied from −10 dB to 0 dB ($\sigma^2_{h_{ij}} = 0.1, \ldots, 1$). The spectrum consisted of 32 parallel frequency bins with independent fading matrices. At each interference level $\sigma_1^2 = \sigma^2_{h_{21}}, \sigma_2^2 = \sigma^2_{h_{12}}$ we randomly picked 25 channels (each comprised of 32 2×2 random matrices). The results of the minimal relative improvement (25) are depicted in Figure 4(b)

$$\Delta_{\min} = \min\{R_1^{\mathrm{NBS}}/R_1^C, R_2^{\mathrm{NBS}}/R_2^C\}. \tag{25}$$

The NBS showed a relative 1.5–3.5 fold gain over the competitive solution, which clearly demonstrates the merits of the method.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## AUTHORS

***Amir Leshem*** (leshema@eng.biu.ac.il) received his B.Sc. degree (cum laude) in mathematics and physics, his M.Sc. degree (cum laude) in mathematics, and his Ph.D. degree in mathematics, all from the Hebrew University, Jerusalem, Israel, in 1986, 1990, and 1998, respectively. From 1998 to 2000, he was with the Faculty of Information Technology and Systems, Delft University of Technology, The Netherlands. From 2000 to 2003, he was director of advanced technologies with Metalink Broadband. Currently, he is also a visiting professor at Delft University of Technology. He is one of the founders of the new school of electrical and computer engineering at Bar-Ilan University, where he is an associate professor and head of the signal processing track. His main research interests include multichannel wireless and wireline communication, brain research, wireless communications and radio-astronomical imaging, set theory, logic, and foundations of mathematics. He is a Senior Member of the IEEE.

***Ephraim Zehavi*** (zehavie1@eng.biu.ac.il) received his B.Sc. and M.Sc. degrees in electrical engineering from the Technion-Israel Institute of Technology in Haifa, in 1977 and 1981, respectively, and his Ph.D. degree in electrical engineering from the University of Massachusetts, Amherst, in 1986. In 1985, he joined QUALCOMM Inc., in San Diego, California. From 1988 to 1992, he was a faculty member at the Department of Electrical Engineering, Technion-Israel Institute of Technology. In 1992, he rejoined QUALCOMM Inc. as a principal engineer. Upon his return to Israel in 1994, he became assistant general manager of Engineering in Qualcomm Israel, Ltd., and later become the general manager. In 2003, he joined the School of Engineering at Bar Ilan, where he is now a professor and the head of the communication track. He is the corecipient of the 1994 IEEE Stephen O. Rice Award and holds more than 40 patents. His main research interests include wireless communications, coding technology, and applications of game theory to communication systems.

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