The Simultaneous Connectivity of Cognitive Networks

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Abstract—In this work we consider the simultaneous connectivity of primary and secondary networks forming a cognitive model. It is assumed that the cognitive model includes guard zones that prevent nodes of the secondary network from being active in the vicinity of primary nodes, to limit interference. Under these assumptions we characterize the region of densities, the transmission radii of the nodes in each of the networks, and the guard zones for which the two networks both have a unique unbounded connected component. We prove that this model is feasible, that is, there exists simultaneous connectivity with a unique unbounded connected component in each of the networks. We also provide necessary and sufficient conditions for the simultaneous connectivity of this cognitive model.

I. INTRODUCTION

In recent decades there has been an ever growing demand for wireless data transmissions. However, the current state of the spectrum allocation cannot adequately respond to these needs. The Federal Communications Commission (FCC), the governing body for spectrum assignment in the US, reports that non-allocated spectrum frequencies are becoming scarce and that new spectrum assignment and usage policies must be implemented in future wireless systems. A closer look at the current state of spectrum usage reveals that almost all the available spectrum is licensed, it is only partially occupied by the licensed users at any given time or location. In order to utilize the unused licensed spectrum and due to the success of wireless networks which communicate over unlicensed bands such as WiFi, Bluetooth, etc., the FCC considers more flexible spectrum usage methods, namely a technology called cognitive radio [1], [2]. Cognitive radio networks serve both types of users: licensed users, who are also known as the primary users, and secondary users who are unlicensed users. Secondary users in cognitive radio networks are equipped with sensing capabilities; utilizing these capabilities they look for communication opportunities that they can take advantage of.

Since overcoming the shortage in non-allocated spectrum frequencies is essential for the prosperity of future communication systems, the cognitive radio paradigm has been studied extensively in recent years. Amongst the numerous works that have been written about this subject are excellent surveys such as [3]–[6] and the works [7]–[22]. These works analyze and optimize the behavior of small cognitive radio communication systems in terms of capacity, power management, outage probability, SINR optimization, etc. The strategies that were developed for small cognitive radio systems are not applicable straightforwardly to networks comprised of large number of users due to the prohibitive complexity of performing necessary tasks such as: channel state acquisition to all users, exact SINR calculation considering all the users in the system, exact power management, etc. This is especially true for mobile ad hoc cognitive systems which do not have a fixed structure.

In their pioneering work [23], Gupta and Kumar analyzed the limiting throughput of ad hoc networks with randomly scattered nodes as the number of nodes grows to infinity. Their work motivated the analysis of the connectivity, capacity and outage probability of large ad hoc networks using techniques from the fields of stochastic geometry and random geometric graphs; these works are surveyed in [24], [25]. One of the most desirable properties of networks in communication theory is connectivity; this property makes it possible to pass information between nodes from all over the network. Ensuring the connectivity of large homogeneous wireless mobile ad hoc network is a challenging task since mobile ad hoc networks do not have a designed backbone but are composed of nodes that are scattered arbitrarily. Moreover, the number of the nodes in the network may not be known exactly, instead only an estimation of this number may be present in advance. These issues encourage taking a probabilistic approach in order to understand and predict the connectivity of large wireless ad hoc networks.

The probabilistic theory that analyzes the connectivity of random graphs on $\mathbb{R}^d$, $d \in \mathbb{N}$ is called continuum percolation [26]. Continuum percolation theory includes the Gilbert disk model which is extremely relevant to the analysis of the connectivity of random networks. The Gilbert disk model is composed of nodes that are generated according to a Poisson point process (PPP); we denote its density by $\lambda$. It is assumed that two nodes are connected if they are within a distance $2\rho$ with each other. A network is said to be connected (or percolates) under continuum percolation models if there exists a unique unbounded connected component in the network. Surprisingly, there exists a critical density $\lambda_c$ for which the network is connected almost surely (a.s.) in the percolation sense for every density that is greater than the critical probability. Further, the network is disconnected a.s. in the percolation sense for every density that is smaller than the critical density, see [26]. Note that in this context, connectivity does not require that all nodes will be connected, but to have a unique unbounded connected component in the network.

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However, since the process is stochastic, and secluded nodes will occur with probability 1, it is reasonable to replace the full connectivity requirement with a connectivity requirement in the percolation sense in order to regain the ability to analyze and foresee the network’s behavior.

The connectivity of cognitive ad hoc systems was analyzed by Ren et. al. in [27] and was also studied in [28]–[31]. These works consider a cognitive model that comprises two interfering networks, the primary network which is composed of primary nodes and the secondary network which is composed of secondary nodes. To limit the interference to the primary network each primary node is protected by a guard zone in which secondary nodes cannot be active. Note that geometrical interference models are well accepted in the Information Theory literature, see for example the protocol model in the celebrated work of Gupta and Kumar [23], or the well known work of Ren et. al. [27] which utilizes similar cognitive interference model; we present the motivation for their acceptance later in the paper. The works [27]–[31] are dedicated to the connectivity analysis of the secondary network with the assumption that the primary network consists of a single hop routing. Contrary to these works, in this paper we study a new cognitive model in which both of the networks are multi-hop ad hoc networks. We analyze in this paper the connectivity of this new model and also redefine the connectivity requirement; we say that this new cognitive model is connected only if both the secondary network and the primary network include an unbounded connected component, a state we call simultaneous connectivity. The results that we present in this paper show that surprisingly, even though the primary network “punctures” the space of the secondary network, the two networks can be simultaneously connected.

Define the simultaneous connectivity region to be the set of all density pairs of the primary and secondary networks, the transmission radii of the primary and secondary nodes and the guard zones radius, for which there is at least one unbounded connected component in the primary network as well as an unbounded connected component in the secondary network a.s. This work aims to analyze the relationship between these terms in order to characterize the simultaneous connectivity region. Our results aim at understanding the connectivity behavior of cognitive ad hoc networks and ensuring that the network is connected with high probability before performing routing protocols. We note that we consider the embedding space of the networks to be \( \mathbb{R}^2 \) but our results can be easily extended to \( \mathbb{R}^d \).

A. Related Works

Many works have extensively studied the connectivity of large-scale homogeneous networks by utilizing percolation theory results, including [23], [32]–[36]. These works consider a one network scenario in which all the nodes in the area of interest belong to the same network.

Several significant papers such as [27]–[31] have considered the connectivity of the secondary network in non-cooperative networks with cognitive nodes. The discussion therein is restricted to models in which the secondary nodes are components of a multi-hop network, but the primary nodes are a part of a single-hop network. Thus, these works do not capture the general connectivity demands of the multi-hop primary network.

Other related works [37]–[41] address a different though related problem of analyzing the throughput of \( n \) primary nodes and \( m \) secondary nodes randomly distributed on the unit square assuming that \( m = n^\beta \), \( \beta > 1 \). In contrast, our work does not assume any prior relation between the densities of the primary and secondary networks.

The analysis presented in this paper aims to provide motivations and insights into generalizations of several current applications of stochastic geometry such as routing [42]–[45], medium-access control [46]–[50] and interference analysis in wireless networks [51]–[55]. For further reading regarding applications of stochastic geometry in the analysis and design of wireless ad hoc networks see for example [53], [56]–[59]. Finally, connectivity is crucial to the promising concept of the internet of things (IoT, see for example [60]–[64]), thus, our analysis is also important for the study of multiple coexisting IoT networks interfering with each other.

B. Contributions of this Work

The main contributions of this work are the following. First, we present a new model for the coexistence of cognitive cognitive networks; this model extends the works [27]–[31] that present a connectivity model only for the secondary network. We also establish the feasibility of the simultaneous connectivity of the cognitive model. This result is novel and counterintuitive, as standard percolation models exhibit a single unbounded connected component. In contrast, our result derives a new model, where each network contains an unbounded connected component. This requires theoretical improvement in terms of the proof technique, since the feasibility of the connectivity of previously studied cognitive models was achieved by reducing the density of the primary network. However, this method cannot be applied to our model, since the primary network must be percolating, therefore, its density cannot be reduced to a value below the critical density. Hence, instead of reducing the density of the primary network which is equivalent to removing nodes (randomly) from the network, we decrease the guard zone between primary and secondary nodes; this translates in practice to power control. We also provide a novel analysis of the necessary conditions for the simultaneous connectivity of the cognitive model. We achieve a significant improvement compared to previous analyses in two aspects: a) the conditions take into account the density of the secondary network b) they hold in any scenario as opposed to previous works in which they hold only for scenarios in which the radius of the guard zone is larger than the transmission range. Thus, our necessary conditions are also adequate for cognitive systems in which the communication rates of the secondary network are lower than the communication rates of the primary network. Further, we provide sufficient conditions for the simultaneous connectivity of the cognitive model. Finally, our results provide for the first time a proof that both random primary and random secondary networks can co-exist and each of them can include an unbounded
connected component. These results lay the foundation for any full future analysis of simultaneous connectivity of cognitive SINR model. They were also recently extended by Sarkar and Haenggi in [65].

C. Organization

The rest of this paper is organized as follows. Section II presents the cognitive network model. In Section III we establish the connectedness of the simultaneous connectivity region, and present the ergodicity of the cognitive model. Section III also establishes that there cannot exist more than one unbounded connected component in each of the networks, and proves the feasibility of the simultaneous connectivity region. Section IV covers the sufficient conditions for the simultaneous connectivity of the cognitive model. Section V presents the necessary conditions for simultaneous connectivity. Section VI presents several extensions to our work, namely, channel availability, node mobility and Cox processes. Finally, Section VII concludes the paper.

II. System Model and Definitions

This section presents the cognitive model and states fundamental definitions and percolation theory results which we apply in our analysis of the connectivity of the cognitive model.

A. The Cognitive Model

In the cognitive model the primary nodes are distributed according to a two-dimensional PPP with density $\lambda_p$. We assume that the transmission range of the primary nodes, $D_i$, is fixed. Similarly, the nodes of the secondary network are distributed according to a PPP with density $\lambda_s$; both PPPs are defined on $\mathbb{R}^2$. Additionally, the PPP of the secondary network is independent of that of the primary network. We also assume that the transmission range of secondary nodes, $d_i$, is fixed.

We next provide several definitions corresponding to the cognitive model.

**Definition 1 (Communication Opportunities of Primary Nodes):** There is a communication opportunity from node $x_i$ to node $x_j$ in the primary network if $\|x_i - x_j\| \leq D_i$, where $\|\cdot\|$ denotes the $L_2$ norm.

**Definition 2 (Active Secondary Nodes):** A secondary node $z_i$ is active if there is no primary node $x$ such that $\|x - z_i\| \leq D_i$. A secondary node is not active, unless its distance from each primary node is greater than $D_i$ (the subscript $f$ stands for the word ‘free’). The guard zones of radii $D_f$ prevent secondary users from being active in the vicinity of primary nodes for following two reasons: 1) To limit the interference to receiving primary nodes. 2) To prevent radio transmissions to receiving secondary nodes that could be severely interfered by primary nodes. Secondary nodes are not active in the first case in order to protect the primary nodes. They are not active in the second case since the interference caused by close primary nodes makes the communication unreliable.

**Definition 3 (Communication Opportunities of Secondary Nodes):** There is a communication opportunity from node $z_i$ to node $z_j$ in the secondary network if the following conditions hold:

1) $\|z_i - z_j\| \leq d_i$,
2) $z_i$ is active,
3) $z_j$ is active.

In this paper we only discuss bidirectional links; that is, we say that there is a link between the nodes $z_i$ and $z_j$ if there exists a communication opportunity from node $z_i$ to node $z_j$ and vice versa. This is typical to most existing communication systems which require feedback.

We note that we consider geometric communication opportunities which ignore the aggregated interference. However, as we discuss in Section II-D our model is highly relevant to practical setups.

**Definition 4:** Let $X_p$ be the set of nodes of the primary network. The connected component of node $x_c \in X_p$ consists of all nodes $x \in X_p$ for which there exists a sequence $x_0 = x_c, x_1, \ldots, x_n = x$ in $X_p$, where for each $0 \leq i \leq n-1$ there is a communication opportunity between $x_i$ and $x_{i+1}$. Additionally, an unbounded connected component of the primary network is a connected component of the primary network which consists of an infinite number of nodes.

The definition of a connected component in the secondary network is similar.

A realization of such a cognitive model is depicted in Figures 1-4 where Fig. 1 depicts the primary network, Fig. 2 includes the guard zone of each primary user and active and passive secondary nodes, and Fig. 3 depicts the active nodes of the secondary network. The largest connected components of primary nodes and active secondary nodes are depicted in Fig. 4.

Finally, since secondary nodes do not affect the connectivity of primary nodes, the connectivity of the primary network is independent of the secondary network, additionally, the connectivity results that are related to the primary network are well established. Thus, this work focuses on percolation in the secondary network for a given density of the primary network, and the parameter space can be reduced by removing $D_i$ assuming that the density of the primary network ensures the connectivity of the primary network. In addition, either $D_f$ or $d_i$ can be set to 1 without loss of generality, since such a re-scaling does not change the model. Hence, the relevant parameters space is only three-dimensional. This is clearly demonstrated by the sufficient and necessary connectivity conditions that are presented in Section IV and Section V, respectively.

B. The Gilbert Disk (Boolean) Model

This section presents the Gilbert disk model on which the analysis of the simultaneous connectivity of the cognitive model defined in Section II-A is based. This definition is used in Section II-C that represents the cognitive model by two Gilbert disk models.

The Gilbert disk model scatters points in $\mathbb{R}^2$ according to a PPP with density $\lambda$ that is defined on $\mathbb{R}^2$. Each disk is associated to a point in the PPP and is assumed to have a
fixed radius $\rho$. In the following we present several definitions related to the Gilbert disk model.

**Definition 5 (Point Process):** Let $\mathcal{B}^2$ be the $\sigma$-algebra of Borel sets in $\mathbb{R}^2$, and let $\mathcal{N}$ be the set of all simple counting measures\(^1\) on $\mathcal{B}^2$. Let $\mathcal{N}$ be the $\sigma$-algebra which is generated by the sets

$$\{ n \in \mathcal{N} : n(A) = k \}, \quad (1)$$

\(^1\)A simple counting measure on $\mathcal{B}^2$ is an integer-valued measure in which the measure of a point is at most one and the measure of every bounded Borel set is finite, for a formal definition see [66, Definition 9.1.II].

where $A \in \mathcal{B}^2$, and $k$ is an integer. A point process $X$ is a measurable mapping from a probability space $(\Omega, \mathcal{F}, P)$ into $(\mathcal{N}, \mathcal{N})$. The distribution of $X$ is denoted by $\mu$ and is defined by $\mu(G) = P(X^{-1}(G))$, for all $G \in \mathcal{N}$. Hereafter, for convenience we refer to $(\mathcal{N}, \mathcal{N})$ as $(\Omega, \mathcal{F})$.

**Definition 6 (Gilbert Disk (Boolean) Model):** Suppose that $X$ is a point process. A Gilbert disk (Boolean) model is composed of point process $X$ and a fixed radius $\rho$ such that each point $x \in X$ is a center of a disk with a fixed radius $\rho$. 

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**Fig. 1.** A realization of a homogeneous network with the parameters set as $\lambda_p = 50 \text{ km}^{-2}$ and $\rho = D_t = 180 \text{ m}$. The black lines indicate the disks that compose the largest connected component.

**Fig. 2.** The guard zone of the primary nodes which appear in Fig. 1 and a realization of the active and passive secondary nodes. The density of the primary network ($\lambda_p$) and the density of the secondary network ($\lambda_s$) are both equal to $50 \text{ km}^{-2}$, and $D_f = 50 \text{ m}$. Green disks depict the guard zone of the primary nodes, blue + indicates active secondary nodes and black x indicates passive secondary nodes.

**Fig. 3.** The active secondary nodes of the realization which appears in Fig. 2 and the parameter $d_t = 220 \text{ m}$. The black lines indicate the disks that compose the largest connected component of the secondary network.

**Fig. 4.** The largest connected components of primary nodes (green), and active secondary nodes (blue) for the realization of the cognitive network depicted in Figures 1-3. The density of the primary network ($\lambda_p$) and the density of the secondary network ($\lambda_s$) are both equal to $50 \text{ km}^{-2}$. Additionally, $D_t = 180 \text{ m}$, $d_t = 220 \text{ m}$ and $D_f = 50 \text{ m}$. 
Since the Boolean model is a generalization of the Gilbert disk model in which each point of the PPP $X$ is a center of a disk with a random radius (see [26, Page 14]), the Gilbert disk model is a special case of a Boolean model with fixed radii. In this paper we assume that $X$ is a PPP with density $\lambda$. We denote the Poisson Gilbert disk model by $(X, \rho, \lambda)$; to distinguish between point in $\mathbb{R}^2$ and $X$ we refer to the points of the PPP $X$ as nodes.

1) Occupied Components: Define $O(z, \rho) \doteq \{ x \in \mathbb{R}^2 : \| x - z \| \leq \rho \}$. Every Poisson Boolean model $(X, \rho, \lambda)$ partitions $\mathbb{R}^2$ into two regions, the occupied region, which we denote by

$$O \doteq \bigcup_{x \in X} O(x, \rho), \quad (2)$$

and the vacant region $\mathcal{V}$ which is the complement of $O$ in $\mathbb{R}^2$. The occupied region consists of the points in $\mathbb{R}^2$ that are covered by at least one disk of radius $\rho$ and a center that belongs to $X$. The vacant region consists of all points in $\mathbb{R}^2$ that are not covered by any disk of radius $\rho$ and a center that belongs to $X$.

Two nodes $x_1, x_2 \in X$ are connected if $O(x_1, \rho) \cap O(x_2, \rho) \neq \emptyset$ (however, in the secondary network we consider only active nodes). The occupied component of $x_c \in X$ is the set $\bigcup_{x \in X_c} O(x, \rho)$ where $X_c$ consists all nodes $x \in X$ for which there exists a sequence $x_0 = x_c, x_1, \ldots, x_n = x$ in $X$, where for each $0 \leq i \leq n - 1$ the disc centered at $x_i$ intersects the disc centered at $x_{i+1}$. The vacant component of a point $v_c \in \mathcal{V}$ consists of all the points $v \in \mathcal{V}$ that have a topological path (in $\mathbb{R}^2$) to $v_c$ that lies in $\mathcal{V}$.

For $A \subseteq \mathbb{R}^2$ let

$$W(A) \doteq \text{the union of all occupied components which have a non-empty intersection with } A. \quad (3)$$

When $A = \{0\}$, we write $W \doteq W(0)$ and we call $W$ the occupied component of the origin. We use similar definitions for the vacant components which we denote by $\mathcal{V}$. Note that only one of the components $\mathcal{V}$ and $W$ can be empty.

2) The Critical Probability: We next define the critical probability of the Gilbert disk model.

Definition 7 (Critical Probability): Let $d(A) \doteq \sup_{x, y \in A} \|x - y\|$ . Denote by $\theta_p(\lambda)$ the probability that the origin is an element of an unbounded occupied component of the Gilbert disk (Boolean) model $(X, \rho, \lambda)$, that is

$$\theta_p(\lambda) \doteq \Pr(d(W) = \infty).$$

The critical density $\lambda_c(2\rho)$ is defined by

$$\lambda_c(2\rho) \doteq \inf\{ \lambda \geq 0 : \theta_p(\lambda) > 0 \}. \quad (4)$$

C. Representation of the Cognitive Model by Gilbert Disk Models

The connectivity of the primary and secondary networks can be studied by representing the two networks by the two independent Gilbert disk models. Nevertheless, because of the guard zones, our connectivity definitions differ from those of a simple Gilbert disk model (see [26]). Since in the Gilbert disk model, two nodes are connected if the distance between them does not exceed $2\rho$, we represent each network by a Gilbert disk model in which $\rho$ is half of the transmission radius. We represent the cognitive network by the following Gilbert disk models $(X_p, d_1/2, \lambda_p)$ and $(X_s, d_2/2, \lambda_s)$, where $(\Omega_p, F_p, P_p)$ and $(\Omega_s, F_s, P_s)$ are the probability spaces of two independent PPPs $X_p$ and $X_s$, respectively. Further, $d_1$ and $d_2$ are the transmission radii in the primary and secondary networks, respectively. In addition, in the secondary network we define an occupied component/region to consist solely of the disks of active secondary nodes. Also, as defined above, in our continuous model the term connected components refers to nodes in the PPPs; the term occupied components refers to the respective discs of the connected components.

D. Interference Model

Since analyzing the connectivity of networks is a complex task, the interference model based on a guard zone was adopted in [27] and [67] even though it does not fully address the aggregated self-interference of the networks and the aggregated interference between the networks. The motivation for accepting this model stems from the fact that close nodes strongly interfere with each other; thus, interference caused by close interferers can be eliminated by applying proper medium access control (MAC) protocols. Assuming that a MAC protocol is implemented and interference caused by close nodes is eliminated, the aggregated interference can be approximated by its expected value. An alternative approach considers the outage probability, i.e., the probability that the aggregated interference is above a certain threshold. These approaches are presented in the monograph [53] and in [68] which consider one network scenario. The cognitive model which depicts cognitive radio networks is analyzed (among others) in [54], [69]. Note that in the cognitive model, the guard zones of the primary nodes prevent secondary nodes from causing strong interference to nearby primary nodes and can be considered as part of the MAC protocol. Finally, the recent work [70] analyzes the relations between physical models in which connectivity opportunities are chosen according to geometrical rules, and protocol models in which connectivity opportunities are chosen according to signal to interference plus noise rules.

III. THE SIMULTANEOUS CONNECTIVITY REGION AND THE UNBOUNDED CONNECTED COMPONENTS

In this section we establish the ergodicity of the cognitive model, and present several results regarding the number of unbounded connected components in the primary and secondary networks. We also define the simultaneous connectivity region and prove that we can always find $D_1 > 0$ such that both the primary and the secondary networks include an unbounded connected component almost surely.

A. The Ergodicity of the Cognitive Model

In order to prove that there is at most one unbounded connected component in the primary network and also in the secondary networks, we define the ergodicity of the cognitive model. Then, we prove that our cognitive model is ergodic.
Definition 8: Let $B^2$ denote the Borel sets of $\mathbb{R}^2$ and let $\Omega$ be a set of simple counting measures on $B^2$. Let $t \in \mathbb{R}^2$, and let $T_t : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by the translation $T_t x = x + t$. $T_t$ then induces the transformation $S_t : \Omega \rightarrow \Omega$ for each $A \in B^2$ through the equation

\[(S_t \omega)(A) = \omega(T_t^{-1} A), \forall \omega \in \Omega. \quad (5)\]

Let $\tilde{\Omega} = \Omega_p \times \Omega_s$, $\tilde{\mathcal{F}} = \mathcal{F}_p \times \mathcal{F}_s$ and $\tilde{\mathcal{P}} = \mathcal{P}_p \times \mathcal{P}_s$, where $(\Omega_p, \mathcal{F}_p, \mathcal{P}_p)$ and $(\Omega_s, \mathcal{F}_s, \mathcal{P}_s)$ are the probability spaces of the PPP of the primary and secondary network, respectively. It follows that $T_t$ induces the transformation $\tilde{T}_t$ on $\tilde{\Omega}$ where

\[\tilde{T}_t = (S_t \omega_p, S_t \omega_s). \quad (6)\]

Definition 9 (Measure Preserving Dynamical Systems): Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and let $T : \Omega \rightarrow \Omega$ be a map that satisfies $\mu(T^{-1} F) = \mu(F)$, for all $F \in \mathcal{F}$; that is $\mu$ is a measure preserving transformation. Then, the quadruple $(\Omega, \mathcal{F}, \mu, T)$ is called a measure preserving (m.p.) dynamical system.

Definition 10 (Mixing): A measure preserving dynamical system $(\Omega, \mathcal{F}, \mu, T)$ is said to be mixing if for all $E, F \in \mathcal{F}$, $\lim_{t \rightarrow \infty} \mu(T^n E \cap F) = \mu(E)\mu(F)$.

Definition 11 (Ergodicity): An event $F \in \mathcal{F}$ is said to be $T$-invariant if $T^{-1} F = F$. The m.p. dynamical system $(\Omega, \mathcal{F}, \mu, T)$ is said to be ergodic if the $\sigma$-algebra of $T$-invariant events is trivial, i.e., any invariant event either has measure of 0 or 1.

Let $\mathcal{G}$ be a group of measure preserving transformations. We say that $\mathcal{G}$ acts ergodically on $(\Omega, \mathcal{F}, \mu, T)$ if $\sigma$-algebra of all the events $F \in \mathcal{F}$ such that $\bigcap \{T \in \mathcal{G} \}(T^{-1} F) = F$ is trivial.

Proposition 3.1: The cognitive model is ergodic with respect to the group $\{\tilde{T}_z(\omega) : z \in \mathbb{Z}^2\}$.

Proof: Since the two Gilbert disk models that compose the cognitive network are generated independently, Proposition 3.1 is a direct result of the mixing property of Poisson point processes [26, proposition 2.6] and [71, Theorem 6.1].

B. The Uniqueness of Unbounded Connected Components

From the ergodicity of the model we deduce that the number of unbounded connected components in each of the networks is constant a.s.

Proposition 3.2: The number of unbounded connected components in the primary network as well as that of the secondary network are constant a.s.

Proof: We prove this proposition by a generalization of the proof of [26, Theorem 2.1]. Let $N_p$, $N_s$ be the random number of unbounded connected components of the primary and secondary networks, respectively. Since the event $\{N_p = k_p, N_s = k_s\}$ is invariant under the group $\{\tilde{T}_z(\omega) : z \in \mathbb{Z}^2\}$, for all $k_p, k_s \geq 0$, by the ergodicity of the cognitive network, the event $\{N_p = k_p, N_s = k_s\}$ is of probability 0 or 1. Therefore, $N_p$, $N_s$ are constant a.s.

Next, we prove that there exists at most one unbounded connected component in each network almost surely.

Theorem 3.3: There is at most one unbounded connected component in the primary network and at most one unbounded connected component in the secondary network a.s.

The proof of this theorem appears in Appendix A. As stated in Appendix A, the proof for the primary network is straightforward. The claim for secondary network is proved by contradiction: Assume that the number of unbounded connected components in the secondary network is finite and greater than one. Then the unbounded connected components of the secondary network can be linked together with positive probability without affecting the number of unbounded connected component of the primary network. The ergodicity of the model concludes the proof. Finally, there cannot be an infinite number of unbounded connected components in the secondary networks by extending a classic continuum percolation result.

C. The Simultaneous Connectivity Region

Definition 12: The simultaneous connectivity region $C$ consists of all 5-tuples $(D_t, d_t, D_s, \lambda_p, \lambda_s)$ such that both the primary and secondary networks include a.s. at least one unbounded connected component.

The simultaneous connectivity region is topologically connected in the space of all the parameters of the problem, i.e., $\lambda_p, \lambda_s, D_t, d_t$ and $D_s$. Previous results (see [27, Theorem 1]) show that the connectivity region is topologically connected in the space of $D_t$ and $\lambda_s$. The connectivity region discussed in [27, Theorem 1] is defined with respect to the connectivity of the secondary network and there are no requirements regarding the connectivity of the primary network.

Proposition 3.4: The simultaneous connectivity region $C$ is topologically connected.

The proof of this theorem is presented in Appendix B.

D. The Feasibility of the Cognitive Model

We next establish in Theorem 3.5 the feasibility of the cognitive model. This feasibility is fundamental to the analysis of the simultaneous connectivity of the cognitive model. Theorem 3.5 is rather counterintuitive since it implies that one can always “puncture” the space $\mathbb{R}^2$ with holes centered at the primary node positions in such a manner that there will still be an unbounded connected component in the secondary network. Note that these “punctures” change the counting measure of the PPP of the active secondary nodes. Also, there is a fundamental difference between the following Theorem 3.5 and Theorem 2 in [27]. The proof of [27, Theorem 2] removes primary nodes with their respective guard zones in order to free space for the secondary nodes, while our proof does not remove primary nodes but only reduces the guard zone around them which in practice translates to power control. This leads to two different modifications of the counting measure of the PPP of the active secondary nodes.

Theorem 3.5: Let $D_t, d_t > 0$ be given. For every $\lambda_p > \lambda_s(D_t)$ and $\lambda_s > \lambda_p(d_t)$ there exists $D_t > 0$ such that there is an unbounded connected component in the primary network and also an unbounded connected component in the secondary network.

The proof of this theorem is presented in Appendix C. Note that this theorem is important since it verifies that the model is “well behaved” in the sense that the connectivity region
is not empty for densities larger than the critical densities of the primary and secondary networks. Additionally, since the secondary nodes cannot harm the connectivity of the primary network, its connectivity is guaranteed by standard percolation results and it is left to analyze connectivity of the secondary network.

IV. SUFFICIENT CONDITIONS FOR SIMULTANEOUS CONNECTIVITY

In this section we present sufficient conditions for the existence of both primary and secondary unbounded connected components. We find these conditions by discretizing the continuous model onto a dependent site percolation model [72], [73]. Our objective is to achieve a discretization that ensures that if there exists an infinite connected component in the discrete site percolation, then an unbounded connected component exists in the continuous model as well. Next we review the site percolation models that we use in this section. Since the exact value of critical probability of the dependent site percolation model is unknown, we consider both the four and eight site percolation models.

A. Site Percolation Models

In site percolation models (see Chapter 2 in [72]), sites are defined to be the squares of the two-dimensional square lattice of $\mathbb{Z}^2$. In the independent setup of this model, the random graph is generated by randomly, identically and independently setting the states of the sites of the two-dimensional square lattice of $\mathbb{Z}^2$ to be occupied (or open)\(^2\) with probability $p$ or vacant (or closed) with probability $1-p$. A component in the site percolation model is connected if every two sites in the component are connected by a path of consecutive occupied sites.

There are two models for site percolation in $\mathbb{Z}^2$: the four neighbors site percolation, and the eight neighbors site percolation. In the four neighbors site percolation model two sites are neighbors if they share a common side, see Fig. 5(a). In the eight neighbors site percolation model two sites are neighbors if they share a common point, see Fig. 5(b). We depict occupied sites by grey filled sites.

Definition 13: The percolation probability $\theta_{\text{site}}(p)$ is the probability that the origin $O$ (or any other vertex, for that matter) is contained in a connected component of an infinite number of sites in the site percolation model.

Definition 14 (Critical probability): The probability $p_c$ is called a critical probability for a site percolation model if $\theta_{\text{site}}(p) = 0$ for $p < p_c$, and $\theta_{\text{site}}(p) > 0$ for $p > p_c$.

A standard way to bound the critical probabilities of the site percolation is the Peierls argument which is depicted in [72, 73]. Our objective is to achieve a discretization that ensures that if there exists an infinite connected component in the discrete site percolation, then an unbounded connected component exists in the continuous model as well. Next we review the site percolation models that we use in this section. Since the exact value of critical probability of the dependent site percolation model is unknown, we consider both the four and eight site percolation models.

Fig. 6 depicts two such circles, one for the four neighbors site percolation model, and the other for the eight neighbors site percolation model. Note that the circle in Fig. 6(b) also bounds the component in the four neighbors site percolation model. However, the circle in Fig. 6(a) does not bound the component in the eight neighbors site percolation model. Also, note that the number of vacant circles of length $n$ around site $x$ is upper bounded by $8n \cdot 7^{n-2}$ for the four neighbors site percolation. For the eight neighbors site percolation this upper bound is $4n \cdot 3^{n-2}$. Thus, using the proof techniques of Theorem 2.2.5 in [72] we can write the following theorems.

Theorem 4.1: For $d = 2$ and the eight neighbors site percolation model there exists $\frac{1}{3} \leq p_s \leq \frac{5}{7}$ such that $\theta_{\text{site}}(p) = 0$ for $p < p_c$ and $\theta_{\text{site}}(p) > 0$ for $p > p_{c_{\text{site}}}$.

Theorem 4.2: For $d = 2$ and the four neighbors site percolation model there exists $\frac{1}{3} \leq p_s \leq \frac{4}{9}$ such that $\theta_{\text{site}}(p) = 0$ for $p < p_c$ and $\theta_{\text{site}}(p) > 0$ for $p > p_{c_{\text{site}}}$.

By Kolmogorov’s zero-one law it follows that if $\psi_{\text{site}}(p)$ be the probability of existence of an infinite connected component in the random grid on $\mathbb{Z}^2$, then $\psi_{\text{site}}(p) = 0$ for $p < p_{c_{\text{site}}}$ and $\psi_{\text{site}}(p) = 1$ for $p > p_{c_{\text{site}}}$.

A dependent site percolation is a site percolation in which the state of a site may depend on the states of other sites. If the state of a site only depends on the states of the sites that are separated by a path of neighboring sites of minimum length $k < \infty$ we say that the model is $k$-dependent (see Fig. 7).
Let \( p \) be the marginal probability for a site to be occupied in a stationary dependent site model. By [72, Theorem 2.3.1], there exists \( p(k) \) such that for \( p > p(k) \) there exists an infinite connected component a.s. It follows that there exists \( p_A(k) \) such that for all \( p > p_A(k) \) there exists an infinite connected component in the \( k \)-dependent site percolation models with four neighbors. Similarly, there exists \( p_S(k) \) such that for all \( p > p_S(k) \) there exists an infinite connected component in the \( k \)-dependent site percolation models with eight neighbors.

### B. The Sufficient Conditions for Simultaneous Connectivity

We proceed to present the sufficient conditions for simultaneous connectivity. As stated above since the exact values of \( p_A(k) \) and \( p_S(k) \) are unknown, we consider both the four and eight site percolation models in Theorem 4.3. Nonetheless, Corollary 4.4 considers upper bounds for which the eight site model yields better results.

Let \( S(t, D_t) \) stand for the area of the intersection between two disks of radius \( D_t \) with centers at distance \( t \). Further, let

\[
\begin{align*}
&f_{Tr,t}(t) \\
= &\mathbb{I}_{\{t \in (0, \ell]\}} \frac{4t}{\ell^4} \left( \frac{\pi \ell^2}{2} - 2t^2 + \frac{t^2}{2} \right) \\
+ &\mathbb{I}_{\{t \in (\ell, \sqrt{2}t]\}} \frac{4t}{\ell^4} \left( \ell^2 \arcsin \left( \frac{2\ell^2 - t^2}{\ell^2} \right) + 2\ell \sqrt{\ell^2 - t^2 - \ell^2 - t^2} - \frac{t^2}{2} \right)
\end{align*}
\]

be the probability density function of the distance \( t \) between two centers of disks, each generated uniformly and independently from a box of side length \( \ell \), which is derived in (54).

\(^{\text{a}}\)

Denote, \( \ell_4 = \frac{d_4}{\sqrt{3}} \), \( \ell_8 = \frac{d_8}{2\sqrt{2}} \).

\[
\begin{align*}
\bar{p}_A(D_t, d_t, \lambda_p, \lambda_s) &\geq \lambda_s d_8^2 e^{-\lambda_p d_t^2} + e^{-\lambda_p D_t^2} \\
&\quad + \left( 1 - \lambda_s d_8^2 e^{-\lambda_p d_t^2} - e^{-\lambda_p D_t^2} \right) e^{-\lambda_p D_t^2} \\
&\quad \cdot \left( 2 - \int_{-\infty}^{\infty} \mathcal{F}_{Tr,t}(t) \cdot e^{\lambda_p [S(t, D_t) - \pi D_t^2]} \, dt \right),
\end{align*}
\]

and

\[
\begin{align*}
\bar{p}_S(D_t, d_t, \lambda_p, \lambda_s) &\geq \lambda_s d_8^2 e^{-\lambda_p d_t^2} + e^{-\lambda_p D_t^2} \\
&\quad + \left( 1 - \lambda_s d_8^2 e^{-\lambda_p d_t^2} - e^{-\lambda_p D_t^2} \right) e^{-\lambda_p D_t^2} \\
&\quad \cdot \left( 2 - \int_{-\infty}^{\infty} \mathcal{F}_{Tr,t}(t) \cdot e^{\lambda_p [S(t, D_t) - \pi D_t^2]} \, dt \right),
\end{align*}
\]

\( \text{Theorem 4.3:} \) Let \( k_4 = \left[ \frac{2\sqrt{5}d_4}{d_t} \right] \) and \( k_8 = \left[ \frac{4\sqrt{2}d_8}{d_t} \right] \). There exist unbounded connected components simultaneously in both the primary and secondary networks if the following conditions hold

\[
\max \left( \frac{\bar{p}_A(D_t, d_t, \lambda_p, \lambda_s)}{p_A(k_4)}, \frac{\bar{p}_S(D_t, d_t, \lambda_p, \lambda_s)}{p_S(k_8)} \right) > 1.
\]

\( \text{Proof:} \)

We discretize the continuous model onto two site models—a four neighbors site model and an eight neighbors site model. In both of the models \((k \ell, m \ell), \quad k, m \in \mathbb{Z} \) are the centers of the squares with side length \( \ell \in \mathbb{R}^+ \). These squares comprise the sites in the site percolation model. Note that while our site percolation model is defined on the grid \((k \ell, m \ell), \quad k, m \in \mathbb{Z} \), there is a one to one mapping from this grid to the grid \((k \ell, m \ell), \quad k, m \in \mathbb{Z} \) that defines the site percolation model.

As previously stated, a site’s neighbors in the four neighbors model are all the sites that have a common side with it. In the eight neighbors model a site’s neighbors are all the sites that have a common side point with it. In both models, a site is said to be occupied if there is at least one secondary user which is active. In order to limit dependency and also to allow each secondary user in a square to communicate with its neighbor, we set \( \ell \) in the four neighbors site model such that \( \sqrt{5}\ell = d_t \), that is \( \ell = \frac{d_t}{\sqrt{5}} \). In this scenario the dependency is of

\[
k_4 = \left[ \frac{2d_t}{\ell} \right] = \left[ \frac{2\sqrt{5}d_t}{d_t} \right]
\]

sites. In the eight neighbors site model we set \( \ell \) such that \( \ell = \frac{d_t}{2\sqrt{2}} \) and

\[
k_8 = \left[ \frac{2d_t}{\ell} \right] = \left[ \frac{4\sqrt{2}d_t}{d_t} \right]
\]

sites.

Let \( p \) be the (marginal) probability that there exists at least one active secondary user in the square \( B = [0, \ell]^2 \). By [72,
Theorem 2.3.1] there exists a probability \( p(k) \), for each of the models, such that for every \( p > p(k) \) there is an infinite connected component in the discrete model a.s. We denote these probabilities by \( p_0(k) \) and \( p_8(k) \).

Denote the event that there are \( m \) secondary nodes in the square \( B \) by \( E_m \). The \( m \) secondary nodes in the square are denoted by \( U_i, i \in \{1, \ldots, m\} \) where the index \( i \) is associated with an arbitrary user. Further, let \( O_t \) be the event in which the secondary user \( U_t \) is active. By definition,

\[
p = \sum_{m=1}^{\infty} \Pr(E_m) \Pr \left( \cup_{i=1}^{m} O_i \right)
\]

\[
\geq \Pr(E_1) \Pr(O_1) + \sum_{m=2}^{\infty} \Pr(E_m) \Pr(O_1 \cup O_2)
\]

\[
= \lambda_s \ell^2 e^{-\lambda_s \ell^2} e^{-\lambda_8 \pi D_t^2}
+ \sum_{m=2}^{\infty} \Pr(E_m) \left[ \Pr(O_1) + \Pr(O_2) - \Pr(O_1 \cap O_2) \right]. \quad (14)
\]

Let \( T \) be the (random) distance between the two secondary nodes \( U_1 \) and \( U_2 \) in the square \( B \). The probability density function \( f_{T,t}(t) \) is derived in (54). We also denote by \( S(t, D_t) \) the intersection area between two disks of radius \( D_t \) with centers at distance \( t \). It follows that

\[
p = \lambda_s \ell^2 e^{-\lambda_s \ell^2} e^{-\lambda_8 \pi D_t^2}
\]

\[
\geq \sum_{m=2}^{\infty} \Pr(E_m) \left[ \Pr(O_1) + \Pr(O_2) - \Pr(O_1 \cap O_2) \right]
\]

\[
= \left( 1 - \Pr(E_1) - \Pr(E_0) \right) \left[ \Pr(O_1) + \Pr(O_2) - \Pr(O_1 \cap O_2) \right]
\]

\[
= \left( 1 - \lambda_s \ell^2 e^{-\lambda_s \ell^2} - e^{-\lambda_s \ell^2} \right) \left[ 2 e^{-\lambda_8 \pi D_t^2} - \Pr(O_1 \cap O_2) \right]
\]

\[
= \left( 1 - \lambda_s \ell^2 e^{-\lambda_s \ell^2} - e^{-\lambda_s \ell^2} \right)
\]

\[
\cdot \left[ 2 e^{-\lambda_8 \pi D_t^2} - \int_{-\infty}^{\infty} f_{T,t}(t) e^{-\lambda_8 (2 \pi D_t^2 - S(t, D_t))} dt \right]
\]

\[
= \left( 1 - \lambda_s \ell^2 e^{-\lambda_s \ell^2} - e^{-\lambda_s \ell^2} \right) e^{-\lambda_8 \pi D_t^2}
\]

\[
\cdot \left[ 2 - \int_{-\infty}^{\infty} f_{T,t}(t) e^{-\lambda_8 [S(t, D_t) - \pi D_t^2]} dt \right]. \quad (15)
\]

A simpler but looser bound than (11) can be derived as the following corollary states.

**Corollary 4.4:** There exists an unbounded connected component in both the primary and secondary networks if the following conditions hold

\[
\lambda_p > D_t^{-2} \lambda_c(1),
\]

\[
\lambda_p < \left( \frac{\lambda_8 D_t^2}{\pi} \right)^{-1} \ln \left( \frac{1 - e^{-\lambda_8 D_t^2/8}}{1 - \left( \frac{1}{3} \right)^2} \right). \quad (16)
\]

**Proof:** Let \( p \) be the (marginal) probability that there exists at least one secondary user in the square \( B = [0, \ell]^2 \) that is active. By the proof of Theorem 4.3

\[
p = \sum_{m=1}^{\infty} \Pr(E_m) \Pr \left( \cup_{i=1}^{m} O_i \right)
\]

\[
\geq \sum_{m=1}^{\infty} \Pr(E_m) \Pr \left( O_1 \right)
\]

\[
= e^{-\lambda_8 \pi D_t^2} \sum_{m=1}^{\infty} \Pr(E_m)
\]

\[
= e^{-\lambda_8 \pi D_t^2} \left( 1 - e^{-\lambda_8 \ell^2} \right)
\]

\[
= e^{-\lambda_8 \pi D_t^2} \left( 1 - e^{-\lambda_s \ell^2} \right). \quad (17)
\]

Additionally, by Sections 2.2 and 2.3 in [72], the condition

\[
1 - p_8(k) < \left( \frac{1}{3} \right)^{(2k_s+1)^2}
\]

is a sufficient condition for the percolation of the eight neighbors site model.

It follows that an equivalent sufficient condition is given by

\[
p_8(k) > 1 - \left( \frac{1}{3} \right)^{(2k_s+1)^2}. \quad (18)
\]

Therefore, a more restrictive requirement is

\[
e^{-\lambda_8 \pi D_t^2} \left( 1 - e^{-\lambda_s \ell^2} \right) > 1 - \left( \frac{1}{3} \right)^{(2k_s+1)^2}. \quad (19)
\]

Eq. (19) can be reorganized in the following manner

\[
e^{-\lambda_8 \pi D_t^2} > 1 - \frac{1}{\left( \frac{1}{3} \right)^{(2k_s+1)^2}}, \quad \lambda_p < \frac{1}{\pi D_t^2} \left( 1 - e^{-\lambda_8 D_t^2} \right) \ln \left( \frac{\lambda_8 D_t^2}{\pi} \right) - 1. \quad (20)
\]

**V. Necessary Conditions for Simultaneous Connectivity**

In this section we prove the necessary conditions for simultaneous percolation in both the primary and secondary networks. These conditions are found by implementing two different methods. The first condition holds only for the scenario where \( 2D_t > d_t \) (Theorem 5.1). It is found by exploiting the fact that there cannot exist both an unbounded vacant component and an occupied component in a Gilbert disk (Boolean) model a.s. The second set of conditions, stated in Theorem 5.2, is found by discretizing the continuous model onto a site percolation model, this model is described in Section IV-A, and bounding the connected component a.s. The second set of conditions is improved in Theorem 5.4 by refining the conditions of a site to be occupied. The necessary conditions of Theorem 5.2 and Theorem 5.4 rescind the restriction that \( 2D_t > d_t \).

**A. The Necessary Conditions**

**Theorem 5.1:** Suppose that \( 2D_t > d_t \), then the following conditions for simultaneous percolation in both networks are necessary:

\[
\lambda_s > d_t^{-2} \lambda_c(1),
\]

\[
\lambda_p > D_t^{-2} \lambda_c(1),
\]
\[ \lambda_p < \left(4D_l^2 - d_l^2\right)^{-1} \lambda_c(1). \]  

The proof of this theorem is a straightforward generalization of Theorem 2.2 in [27].

When the condition that \(2D_l > d_l\) does not hold, the above theorem does not apply; we provide necessary conditions for the general case in which \(2D_l\) is not necessarily greater than \(d_l\). These conditions for simultaneous connectivity are obtained by discretization onto a site percolation, that in which each site has eight neighbors, this model is depicted in Section IV-A. These conditions also explore the relationship between the densities of the primary and secondary networks under the simultaneous percolation regime.

**Theorem 5.2:** Let \( n_p = \left[\frac{\sqrt{D_l}}{D_l}\right]^2 \). Denote by \( p_s\) the critical probability of site percolation with eight neighbors. Then the following conditions for simultaneous percolation in both networks are sufficient:

\[ \lambda_s > d_l^{-2} \lambda_c(1), \]
\[ \lambda_p > D_l^{-2} \lambda_c(1), \]
\[ \lambda_p < \frac{n_p}{d_l^2} \ln \left(1 - \left(1 - e^{-\lambda_s d_l^2}\right)^{-1/n_p} \left(1 - e^{-\lambda_s d_l^2} - p_s\right)^{1/n_p}\right). \]  

**Proof:** As previously, the first two conditions of Eq. (22) are rudimentary conditions for the connectivity of the primary and the secondary networks. To establish the third condition of Eq. (22) we discretize the continuous model onto a discrete model. We choose a model in which non-occurrence of a percolation dictates that there cannot be an unbounded connected component in the secondary network.

We discretize the continuous model onto an eight neighbors site model in the following manner. Partition \( \mathbb{R}^2 \) into squares of side length \( d_l \), i.e., the squares \( s(m,k) = \left[\frac{-d_l}{2} + m \cdot d_l, \frac{d_l}{2} + m \cdot d_l\right] \times \left[\frac{-d_l}{2} + k \cdot d_l, \frac{d_l}{2} + k \cdot d_l\right] \) where \( m, k \in \mathbb{Z} \). Partition each square into \( n_p = \left\lfloor \frac{d_l}{D_l\sqrt{2}} \right\rfloor^2 \) sub-squares, each of side length \( \frac{d_l}{n_p} \).

We say that a square \( s(m,k) \) is vacant if it does not contain any connected node or if every sub-square of \( s(m,k) \) contains at least one primary user. The probability that a square is vacant is the probability of the union of the two events

\[ \left(1 - e^{-\lambda_s d_l^2}\right) \left(1 - e^{-\lambda_p \frac{d_l^2}{n_p}}\right)^{n_p} + e^{-\lambda_s d_l^2}. \]  

Let \( p_s \) be the critical probability of the eight neighbors site percolation model. By discrete percolation if the probability for a site to be vacant is greater than \( 1 - p_s \), then every connected component is finite a.s. For the eight neighbors site model this implies that when

\[ \left(1 - e^{-\lambda_s d_l^2}\right) \left(1 - e^{-\lambda_p \frac{d_l^2}{n_p}}\right)^{n_p} + e^{-\lambda_s d_l^2} > 1 - p_s \]  

every connected component in the secondary network is bounded a.s. Equivalently, an unbounded connected component may exist in the secondary network only if

\[ \left(1 - e^{-\lambda_s d_l^2}\right) \left(1 - e^{-\lambda_p \frac{d_l^2}{n_p}}\right)^{n_p} + e^{-\lambda_s d_l^2} < 1 - p_s. \]  

Further algebra yields the following condition

\[ \lambda_p < -\frac{n_p}{d_l^2} \ln \left(1 - \left(1 - e^{-\lambda_s d_l^2}\right)^{-1/n_p} \left(1 - e^{-\lambda_s d_l^2} - p_s\right)^{1/n_p}\right). \]  

By applying the lower bound \( \frac{1}{3} \leq p_s \) (see [72, Chapter 2.2]) to Theorem 5.2 we obtain the following corollary.

**Corollary 5.3:** Let \( n_p = \left[\frac{\sqrt{D_l}}{D_l}\right]^2 \). Then the following conditions for simultaneous percolation in both networks are necessary:

\[ \lambda_s > d_l^{-2} \lambda_c(1), \]
\[ \lambda_p > D_l^{-2} \lambda_c(1), \]
\[ \lambda_p < \frac{n_p}{d_l^2} \ln \left(1 - \left(1 - e^{-\lambda_s d_l^2}\right)^{-1/n_p} \left(2 - 3 e^{-\lambda_s d_l^2}\right)^{1/n_p}\right). \]  

In the next theorem establish a stricter necessary condition compared to that of Theorem 5.2. We achieve this by finding additional events that cause a site to be vacant in such a way that prohibits simultaneous connectivity.

**Theorem 5.4:** Let \( n_p = \left[\frac{\sqrt{D_l}}{D_l}\right]^2 \). Denote by \( p_s\) the critical probability of site percolation with eight neighbors. Then the following conditions for simultaneous percolation in both networks are necessary:

\[ \lambda_s > d_l^{-2} \lambda_c(1), \]
\[ \lambda_p > D_l^{-2} \lambda_c(1), \]
\[ \lambda_p < -\frac{n_p}{d_l^2} \ln \left(1 - \left(1 - p_s\right)^{1/n_p} - e^{-\lambda_s \frac{d_l^2}{n_p}}\right)^{n_p} \left(1 - e^{-\lambda_s \frac{d_l^2}{n_p}}\right)^{-1}. \]  

**Proof:** In this proof we discretize the continuous cognitive model onto the discrete model that is depicted in the proof of Theorem 5.2. We provide another set of necessary conditions by defining a site \( s(m,k) \) to be vacant if one of the following events occur for each of its sub-squares

1) There is no secondary node in the sub-square.
2) There is at least one secondary node and one primary node in the sub-square.

Under this condition, the probability of a site to be vacant is

\[ e^{-\lambda_s \left(\frac{d_l^2}{n_p}\right)} + \left(1 - e^{-\lambda_s \left(\frac{d_l^2}{n_p}\right)}\right)^{n_p} \left(1 - e^{-\lambda_s \left(\frac{d_l^2}{n_p}\right)}\right)^{-1}. \]  

Hence, an unbounded connected component may exist in the secondary network only if

\[ e^{-\lambda_s \left(\frac{d_l^2}{n_p}\right)} + \left(1 - e^{-\lambda_s \left(\frac{d_l^2}{n_p}\right)}\right)^{n_p} < 1 - p_s. \]  

Thus,

\[ \lambda_p < -\frac{n_p}{d_l^2} \ln \left(1 - \left(1 - p_s\right)^{1/n_p} - e^{-\lambda_s \left(\frac{d_l^2}{n_p}\right)}\right)^{n_p} \left(1 - e^{-\lambda_s \left(\frac{d_l^2}{n_p}\right)}\right)^{-1}. \]
is a necessary condition for the simultaneous connectivity. By applying the lower bound \( \frac{1}{3} \leq p_8 \) (see [72, Chapter 2.2]) to Theorem 5.4 we obtain the following corollary.

**Corollary 5.5:** Let \( n_p \doteq \left[ \frac{S_d}{2D_t} \right]^2 \), then the following conditions for simultaneous percolation in both networks are necessary:

\[
\begin{align*}
\lambda_s &> d_i^{-2} \lambda_s(1), \\
\lambda_p &> D_t^{-2} \lambda_s(1), \\
\lambda_p < -\frac{n_p d_i^2}{d_t^2} \ln \left[ 1 - \left( \frac{2}{3} \right)^{1/n_p} - e^{-\lambda_s \left( \frac{d_i^2}{d_t^2} \right)} \left( 1 - e^{-\lambda_s \left( \frac{d_i^2}{d_t^2} \right)} \right)^{-1} \right].
\end{align*}
\]

(B) Comparison of the Necessary Conditions

We next compare the necessary conditions of Theorem 5.1, Corollary 5.3 and Corollary 5.5. First, note that Theorem 5.1 only holds when \( 2D_t > d_i \) and is not affected by the value of \( \lambda_s \), hence, it is not relevant for small values of \( D_t \). Additionally, Corollary 5.5 refines the results of Corollary 5.3 as noted above.

Figure 8 depicts the necessary conditions for simultaneous connectivity. It includes four plots of the necessary conditions of \( \lambda_p \) as a function of \( \lambda_s \). It displays four regimes of the relations between the three necessary conditions. Figure 8(a) describes the region \( 2D_t < d_i \) in which Theorem 5.1 does not apply. Figure 8(b) refers to the region \( 2D_t < d_i \) in which Theorem 5.1 does apply, however, \( 2D_t \approx d_i \) and so the necessary conditions of Theorem 5.1 are less restrictive than those of Corollary 5.3 and Corollary 5.5. Figure 8(c) includes a slightly larger value of \( D_t \). For these values, Corollary 5.5 is tighter than Theorem 5.1 for small values of \( \lambda_s \). For other values of \( \lambda_s \), Theorem 5.1 is tighter than Corollary 5.5. Finally, Figure 8(d) presents a scenario in which Theorem 5.1 is tighter than Corollary 5.5 for all values of \( \lambda_s \). Note that in this case, there is no simultaneous connectivity of the two networks for the chosen value of \( D_t \).

Figure 9 depicts the necessary conditions of \( \lambda_p \) as a function of \( D_t \) for \( 2D_t < d_i \). It includes two figures, one for each of two values of \( \lambda_s \): (a) 13.61\,km\(^{-2}\) (b) 217.77\,km\(^{-2}\). In this scenario because \( 2D_t < d_i \), Theorem 5.1 does not apply. We can see that as \( \lambda_s \) increases, the gap between the results of Corollary 5.3 and Corollary 5.5 reduces. Furthermore, the maximal necessary values of \( \lambda_p \) as a function of \( D_t \) is a nonincreasing function. The step-function nature of the plot arises from the ceiling operation in the definition of \( n_p \) in both Corollary 5.3 and Corollary 5.5.

Figure 10 portrays the necessary conditions of \( \lambda_p \) as a function of \( D_t \) for \( 2D_t > d_i \). It includes three plots in each figure, in addition to the critical probability line of the primary network, for two different values of \( \lambda_s \): (a) 40.83\,km\(^{-2}\) (b) 680.53\,km\(^{-2}\). Here because \( 2D_t > d_i \), Theorem 5.1 applies. We can see again that as \( \lambda_s \) increases, the gap between the results of Corollary 5.3 and Corollary 5.5 reduces. Furthermore, as before the maximal necessary values of \( \lambda_p \) as a function of \( D_t \) is a nonincreasing step function. For this region, as \( D_t \) increases and is larger from \( d_i/2 \) the better are the conditions of Theorem 5.1. This phenomenon occurs since the condition upper bounding the primary network’s density is expressed in Theorem 5.1 as a function of \( 1/2D_t \) asymptotically.

VI. RELEVANT EXTENDED MODELS

The results of this paper can be extended straightforwardly to the following scenarios. The first scenario that can be easily presented as an extension of our work is a cognitive model in which each primary node has an available channel for communication with some probability \( \alpha_p \). In this case, we represent the primary network by the Gilbert disk model \((\lambda_p, D_t/2, \alpha_p, \lambda_p)\), it follows that the results presented in this article hold with the adjustment of the density of the primary nodes to be \( \alpha_p \lambda_p \). The case where each secondary node has an available channel for communication with some probability \( \alpha_s \) is similar. Our results are also readily extended to mobile cognitive networks in which the nodes of one or both of the networks move randomly and arbitrarily. In this case the new locations of the nodes are also homogeneous PPPs and thus our results still hold. Finally, the results presented in this paper are also relevant to the analysis of Cox process which are also known as Poisson hole process. Assuming that the holes can be bounded from the inside and outside by bounded circles, we can use the results presented in this paper to establish connectivity results for Cox processes.

VII. CONCLUSION

In this paper we presented several properties of the simultaneous connectivity of the cognitive network model. We proved that there cannot exist more than one unbounded connected component in each of the networks. Moreover, we presented sufficient as well as necessary conditions for the simultaneous connectivity of the cognitive model. Furthermore, we argued that for each pair of densities greater than the critical density without inter-network interference, there exists a small enough guard zone such that there exist unbounded connected components in each of the networks. We hope that these results will motivate further discussion on applications and performance of such cognitive ad-hoc networks.

APPENDIX A

PROOF OF THEOREM 3.3

**Proof of Theorem 3.3:** Since the secondary nodes transmit only if they do not cause interference to the primary nodes, we can use [26, Theorem 3.6] to conclude that the number of unbounded connected components in the primary network, i.e., \( N_p \), is at most one a.s. We next prove by contradiction that the number of unbounded connected components in the secondary network is at most one a.s. Note that the proof generalizes the proof of [26, Proposition 3.3] and the proof of [27, Lemma 2].

Assume towards contradiction that the number of unbounded connected components in the secondary network is greater than one, i.e., \( N_s \geq 2 \). Suppose that \( N_s \) is a finite number such that \( N_s \geq 2 \). By Proposition 3.2 it suffices to contradict this assumption by proving that with a positive probability all \( N_s \) unbounded connected components can be
linked together (by adding secondary nodes and deleting primary nodes) without affecting the number of unbounded connected components $N_p$, in the primary network a.s.

By assumption $N_s$ is finite, therefore, there exists $n \in \mathbb{N}$ such that the box $B = [-n, n]^2$ includes at least one secondary node from each unbounded connected component. For each $A \subset \mathbb{R}^2$, let $C_s[A]$ denote the occupied region formed by the secondary nodes in $A$, that is, $C_s[A] = \bigcup_{z_i \in A} O(z_i, d_t/2)$. For a box $B$ and some $\epsilon > 0$ we define the event $A(B, \epsilon)$ by:

$$A(B, \epsilon) = \{\tilde{\omega} \in \tilde{\Omega} : d(U, B) \leq d_t/2 - \epsilon \text{ for any unbounded occupied component } U \text{ in } C_s[B^c]\}. \quad (33)$$

This event includes all the events in which for each occupied component in the secondary network outside the square $B$ there exists a secondary node which is within a distance of $d_t/2 - \epsilon$ from $B$.

Partition the square $B$ into square cells with side length $a = a(B, \epsilon) > 0$, and let $S_a = \{S_1, \ldots, S_K\}$ be the collection of all cells that are adjacent to the boundary of the box $B$. For every box $B$ and $\epsilon > 0$ we can find $a = a(B, \epsilon) > 0$ and $\eta = \eta(a) > 0$ such that for any node $z$ in the secondary network...
such that $z \notin B$ and $d(z, B) \leq \frac{d_t}{2} - \epsilon$ there exists a square $S \in S_a$ such that $\sup_{z \in S} d(z, z_B) \leq d_t - \eta$. It follows that if we place a secondary node in each of the boundary cells $S_a$ and if there are no primary nodes in $B^c$ then every unbounded occupied component $U$ in $C_s[B^c]$ such that $d(U, B) \leq \frac{d_t}{2} - \epsilon$ is connected to a node in $S_a$ as in Fig. 11.

Let $A(a, \eta)$ be the event where there exists at least one secondary node in each square in $S_a$, and let $A_p(B)$ be the event where there are no primary nodes in the larger box $B$. We get that

$$\Pr(A(B, \epsilon) \cap A(a, \eta) \cap A_p(B) \cap \{N_p = k\}) = \Pr(A(B, \epsilon)) \Pr(A(a, \eta)) \Pr(A_p(B) | A(B, \epsilon)) \cdot \Pr(N_p = k | A(B, \epsilon)) \cap A(a, \eta) \cap A(B)) \cdot \Pr(N_p = k | A(B, \epsilon)) \cap A(a, \eta) \cap A(B)). \quad (34)$$

As in the proof of [26, Proposition 3.3] and the proof of [27, Lemma 2], there exists a box $B$, and constants $\epsilon, a, \eta > 0$ such that $\Pr(A(B, \epsilon)) \cdot \Pr(A(a, \eta)) > 0$. Additionally,

$$\Pr(A_p(B) | A(B, \epsilon)) \geq \Pr(A_p(B)) > 0 \quad (35)$$

where the rightmost inequality holds since the box $B$ is bounded.

Since $\Pr(A(B, \epsilon)) \Pr(A(a, \eta)) \Pr(A_p(B) | A(B, \epsilon)) > 0$, by the law of total probability if $\Pr(N_p = k) = 1$ then $\Pr(N_p = k | A(B, \epsilon)) \cap A(a, \eta) \cap A(B)) > 0$, and if $\Pr(N_p = k) = 0$, then $\Pr(N_p = k) = 0$. Hence,

$$\Pr(N_p = k | A(B, \epsilon)) \cap A(a, \eta) \cap A(B)) > 0.$$

Fig. 9. The necessary conditions for two values of $\lambda$: (a) $13.61\text{km}^{-2}$ and transmission ranges of $D_t = 210\text{m}$ and $d_t = 230\text{m}$. Note that Theorem 5.1 does not hold for scenarios (a) or (b).

Fig. 10. The necessary conditions for two values of $\lambda$: (a) $40.83\text{km}^{-2}$ and (b) $680.53\text{km}^{-2}$ and transmission ranges of $D_t = 210\text{m}$ and $d_t = 230\text{m}$.

Fig. 11. Linking two occupied components of the secondary network.
$k|A(B, e) \cap A(a, \eta) \cap A(\overline{B})| = 0$. Therefore by Proposition 3.2 there are either 0 or 1 unbounded connected components in the primary network and either 0 , 1 or $\infty$ unbounded connected components in the secondary network.

It remains to be proven that there cannot be an infinite number of unbounded connected components in the secondary network. By implementing [26, Lemma 3.2] as in the proof of [26, Theorem 3.6] it follows that there cannot be an infinite number of unbounded occupied components in the secondary networks. We note that [26, Theorem 3.6] it follows that there cannot be an infinite network. By implementing [26, Lemma 3.2] as in the proof of (26, p. 66-68) hold for the secondary network as well.

**Appendix B**

This appendix includes the proof of the topological connectivity of the connectivity region, i.e., Proposition 3.4.

The proof is comprised of several steps that utilize the definition of the following connectivity regions.

**Definition 15:** We define the following simultaneous connectivity regions:

- The simultaneous connectivity region $C(D_t, d_t, D_t)$ consists of all pairs of densities $(\lambda_p, \lambda_s)$ such that both the primary and secondary networks include a.s. at least one unbounded connected component for a given vector of parameters $(D_t, d_t, D_t)$.
- The simultaneous connectivity region $C(d_t, D_t)$ consists of all triples $(D_t, \lambda_p, \lambda_s)$ such that both the primary and secondary networks include a.s. at least one unbounded connected component for a given vector of transmission radii $(D_t, d_t)$.
- The simultaneous connectivity region $C(D_t)$ consists of all 4-tuples $(d_t, D_t, \lambda_p, \lambda_s)$ such that both the primary and secondary networks include a.s. at least one unbounded connected component for a given vector of transmission radius $D_t$.

**Outline of the proof of Proposition 3.4:** We prove Proposition 3.4 sequentially. First we prove in Lemma B.1 that the region $C(D_t, d_t, D_t)$ is topologically connected for each vector of parameters $(D_t, d_t, D_t)$ by extending the proof of [27, Theorem 1] for the connectivity of the primary network. Then, we establish in the first part of Lemma B.2 the topological connectedness of the simultaneous connectivity region $C(D_t, d_t)$ for each vector of parameters $(D_t, d_t)$. This topological connectedness follows from the fact that reducing $D_t$ only improves the chance of the occurrence of topological connectedness. The second part of Lemma B.2 establishes the topologically connectedness of the simultaneous connectivity region $C(D_t)$ for each value of the parameter $D_t$. This topological connectedness follows from the fact that increasing $d_t$ only improves the chance of the occurrence of topological connectedness. The last step of the proof finalizes the proof of Proposition 3.4.

**Lemma B.1:** The simultaneous connectivity region $C(D_t, d_t, D_t)$ is topologically connected for each vector of parameters $(D_t, d_t, D_t)$.

**Proof:**

The proof generalizes the proof of [27, Theorem 1]. Note that unlike [27], we need to ensure the connectivity of both the primary and the secondary networks. For ease of notation we refer to $C(D_t, d_t, D_t)$ as $\hat{C}$.

Let $(\lambda_{p1}, \lambda_{s1})$ and $(\lambda_{p2}, \lambda_{s2})$ be two pairs of densities in the simultaneous connectivity region. We assume without loss of generality that $\lambda_{s1} \leq \lambda_{s2}$, and prove that there is a topological path from $(\lambda_{p1}, \lambda_{s1})$ to $(\lambda_{p2}, \lambda_{s2})$ which resides in $\hat{C}$. We consider the topological path that is constructed by the horizontal segment which starts at $(\lambda_{p1}, \lambda_{s1})$ and ends at $(\lambda_{p1}, \lambda_{s2})$ and the vertical segment which starts at $(\lambda_{p1}, \lambda_{s2})$ and ends at $(\lambda_{p2}, \lambda_{s2})$. We now distinguish between two cases: case (a) in which $\lambda_{p1} \leq \lambda_{p2}$, and case (b) in which $\lambda_{p1} \geq \lambda_{p2}$.

We present the proof for case (b); case (a) follows similarly.

As mentioned above, we choose the topological path that consists of two segments. We now prove that each of these segments lies in the simultaneous connectivity region $\hat{C}$. First, we show that the segment $(\lambda_{p1}, \lambda_{s1})$ where $\lambda_{s1} \leq \lambda_{s2}$ lies in $\hat{C}$. Since the secondary nodes transmit only if they do not interfere with the transmissions of primary nodes, and since the pair $(\lambda_{p1}, \lambda_{s1})$ lies in $\hat{C}$, the primary network includes a.s. an unbounded connected component for each pair of densities in the segment $(\lambda_{p1}, \lambda_{s1})$. Further since $\lambda_{s1} \leq \lambda_{s2}$, it follows by superposition techniques (see [27, Theorem 1] and [26, p. 11]) that the secondary network has an unbounded connected component as well. Therefore, the segment $(\lambda_{p1}, \lambda_{s1})$ where $\lambda_{s1} \leq \lambda_{s} \leq \lambda_{s2}$ is in $\hat{C}$.

Second, we show that the segment $(\lambda_{p2}, \lambda_{s2})$ where $\lambda_{p2} \leq \lambda_{p1} \leq \lambda_{p2}$ lies in $\hat{C}$. One can argue by superposition techniques that for each of the densities $(\lambda_{p2}, \lambda_{s2})$ such that $\lambda_{p2} \leq \lambda_{p} \leq \lambda_{p1}$, the primary network includes an unbounded connected component a.s. Consequently, by Definition 3 of secondary node communication opportunities, if $(\lambda_{p1}, \lambda_{s2})$ lies in $\hat{C}$, it follows that $(\lambda_{p}, \lambda_{s2})$ lies in $\hat{C}$.

**Lemma B.2:** The following statements hold for the cognitive model:
(i) The simultaneous connectivity region \( C(D_f, D_l) \) is topologically connected for each vector of parameters \((D_f, D_l)\).

(ii) The simultaneous connectivity region \( C(D_l) \) is topologically connected for each value of the parameter \( D_l \).

**Proof:**

(i) Let \((D_{l1}, \lambda_{p1}, \lambda_{s1})\) and \((D_{l2}, \lambda_{p2}, \lambda_{s2})\) be in \( C(D_l, D_l) \). We show that there exists a topological path in \( C(D_l, D_l) \) which connects these two triples. Suppose without loss of generality that \( D_{l1} \geq D_{l2} \). We prove that \((D_l, \lambda_{p1}, \lambda_{s1}) \in C(D_l, D_l)\) for each \( D_{l2} \leq D_l \leq D_{l1} \). Let \((\omega_{p1}, \omega_{s1})\) be a realization of the cognitive model with a pair of densities \((\lambda_{p1}, \lambda_{s1})\) and a guard zone radius \( D_{l1} \) such that there is at least one unbounded connected component in both the primary and secondary networks. For each such realization, decreasing the guard zone allows more secondary nodes to be active, or at minimum, all the previously active secondary nodes are still active. Additionally, decreasing the guard zone does not affect the primary network. It follows that decreasing the guard zone radius does not reduce the number of unbounded connected components in each of the networks. Therefore, if \((D_{l1}, \lambda_{p1}, \lambda_{s1}) \in C(D_l, D_l)\) then \((D_{l1}, \lambda_{p1}, \lambda_{s1}) \in C(D_l, D_l)\) for each \( D_{l2} \leq D_l \leq D_{l1} \). Since \((D_{l2}, \lambda_{p1}, \lambda_{s1})\) and \((D_{l2}, \lambda_{p2}, \lambda_{s2})\) are in \( C(D_l, D_l)\), by Lemma B.1 the two triples are connected in \( C(D_l, D_l) \).

(ii) Let \((d_{l1}, D_{l1}, \lambda_{p1}, \lambda_{s1})\) and \((d_{l2}, D_{l2}, \lambda_{p2}, \lambda_{s2})\) be in \( C(D_l) \). We show that there exists a topological path in \( C(D_l) \) that connects these two 4-tuples. Suppose without loss of generality that \( d_{l1} \geq d_{l2} \). We prove that \((d_{l1}, D_{l1}, \lambda_{p1}, \lambda_{s1}) \in C(D_l)\) for each \( d_{l1} \leq d_l \leq d_{l2} \). Let \((\omega_{p1}, \omega_{s1})\) be a realization of the cognitive model with a pair of densities \((\lambda_{p1}, \lambda_{s1})\), a guard zone radius \( D_{l1} \) and a transmission radius \( d_{l1} \) such that there is at least one unbounded connected component in both the primary and secondary networks. For this realization, when increasing the transmission radius of the secondary nodes, all the previously connected nodes are still connected, and we can only connect more nodes in the secondary network without affecting the primary one. It follows that increasing the transmission radius of the secondary nodes does not reduce the number of unbounded connected components in each of the networks. Therefore, if \((d_{l1}, D_{l1}, \lambda_{p1}, \lambda_{s1}) \in C(D_l)\) then \((d_{l2}, D_{l2}, \lambda_{p1}, \lambda_{s1}) \in C(D_l)\) for each \( d_{l1} \leq d_l \leq d_{l2} \). Since \((D_{l2}, D_{l1}, \lambda_{p1}, \lambda_{s1})\) and \((D_{l2}, D_{l2}, \lambda_{p2}, \lambda_{s2})\) are in \( C(D_l)\), by part (i) of this lemma the two 4-tuples are connected in \( C(D_l) \).

**Proof of Proposition 3.4:** Let \((D_{l1}, d_{l1}, D_{l1}, \lambda_{p1}, \lambda_{s1})\) and \((D_{l2}, d_{l2}, D_{l2}, \lambda_{p2}, \lambda_{s2})\) be in \( C \). We show that there exists a topological path in \( C \) that connects these two 5-tuples. Suppose without loss of generality that \( D_{l1} \leq D_{l2} \). We can prove similarly to the proof of the second part of Lemma B.2 that \((D_l, d_{l1}, D_{l1}, \lambda_{p1}, \lambda_{s1}) \in C \) for each \( D_{l1} \leq D_l \leq D_{l2} \). Therefore, if \((D_{l1}, d_{l1}, D_{l1}, \lambda_{p1}, \lambda_{s1}) \in C \) then \((D_{l2}, d_{l1}, D_{l1}, \lambda_{p1}, \lambda_{s1}) \in C \) for each \( D_{l1} \leq D_l \leq D_{l2} \). Since \((D_{l2}, d_{l1}, D_{l1}, \lambda_{p1}, \lambda_{s1})\) and \((D_{l2}, d_{l2}, D_{l2}, \lambda_{p2}, \lambda_{s2})\) are in \( C \), by part (ii) of Lemma B.2 the two 5-tuples are connected in \( C \).

**Appendix C**

**Proof of Theorem 3.5**

The proof of Theorem 3.5 utilizes the well established theory of the bond percolation model, a thorough description of this model is available in numerous works, among them are [73, Chapter 1] and [72, Chapter 2].

Before proceeding to prove Theorem 3.5 we present several definitions and results regarding the connectivity probabilities that we use in the proof of this theorem.

### A. Crossings Probabilities

**Definition 16:** A topological path \( \nu \) of points in \( \mathbb{R}^2 \) is said to be occupied if it lies in the occupied region\(^a\), i.e., \( \nu \subset O \).

An occupied path \( \nu \) is an occupied left-right (LR) crossing of the rectangle \([0, l_1] \times [0, l_2]\) if there exists a segment \( \tau \) of \( \nu \) which is contained in the rectangle \([0, l_1] \times [0, l_2]\) and it also intersects the left and right boundaries of the rectangle \([0, l_1] \times [0, l_2]\). We define an occupied top-bottom (TB) crossing of the rectangle in a similar manner.

Let \( \sigma(l_1, l_2, \lambda, LR) \) be the LR crossing probability of the rectangle \([0, l_1] \times [0, l_2]\); that is, the probability of generating a PPP \( X \) such that there exists an occupied LR crossing of the rectangle \([0, l_1] \times [0, l_2]\). Also, let \( \sigma(l_1, l_2, \lambda, TB) \) denote the TB crossing probability of the rectangle \([0, l_1] \times [0, l_2]\). The critical probability \( \lambda_{c} \) of a Gilbert disk model has a strong tie to the crossing probability as the next result shows.

Suppose that \((X, \lambda, \rho)\) is a Gilbert disk model with a bounded \( \rho \). Then by [26, Corollary 4.1] it follows that for all \( k \geq 1 \),

\[
\lim_{n \to \infty} \sigma((kn, n), \lambda, LR) = \begin{cases} 1, & \text{if } \lambda > \lambda_{c}(2\rho) \\ 0, & \text{if } \lambda < \lambda_{c}(2\rho). \end{cases} \tag{36}
\]

That is, the limit as \( n \) grows to infinity of the probability of the existence of LR crossing in the rectangle \([0, kn] \times [0, n]\) for a PPP with density \( \lambda \) tends to one if \( \lambda \) is greater than the critical probability and zero if \( \lambda \) is smaller than the critical probability. By symmetry, a similar result holds for the TB crossings, that is,

\[
\lim_{n \to \infty} \sigma((n, kn), \lambda, TB) = \begin{cases} 1, & \text{if } \lambda > \lambda_{c}(2\rho) \\ 0, & \text{if } \lambda < \lambda_{c}(2\rho). \end{cases} \tag{37}
\]

The case of \( \lambda = \lambda_{c}(2\rho) \) was recently investigated in [74] in the two dimensional setup. It was concluded in [74] that for every \( k > 0 \) there exists \( c_0 = c(k) > 0 \) such that \( 0 < \sigma((kn, n), \lambda_{c}, LR) < 1 - c_0 \) for every \( n \geq 1 \).

### B. Proof of Theorem 3.5

**Proof:**

Since the primary network includes an unbounded connected component a.s., we proceed to consider the connectivity of the secondary network. For the sake of this proof we discretize the continuous model onto a discrete model \( \mathcal{L} \). We then analyze this discretized model using a discrete percolation model called “bond percolation” (see [73, Chapter 1] and [72, Chapter 2]) to prove that there exists \( D_f > 0 \) such that

\(^a\)See (2).
the cognitive model is simultaneously connected a.s. Next we describe the discrete model $\mathcal{L}$ that is depicted in Fig. 12.

Fig. 12. Discretization of the continuous model. The edge $e$ is open since the three crossings exist.

Set $\ell \in \mathbb{R}^+$ such that $\ell > 2(2D_i + d_t)$ and place the vertices of the planar graph in $(lk, \ell m)$ where $k, m \in \mathbb{Z}$. Each vertex $(lk, \ell m)$ is connected to the vertices $(l(k \pm 1), \ell m), (lk, \ell (m \pm 1))$.

The set of all edges is denoted by $E$. Denote the middle point of the edge $e \in E$ by $(x_e, y_e)$. An edge $e \in E$ is said to be open if the following conditions hold:

1. There is an LR occupied crossing of secondary nodes in the rectangle $[x_e - \frac{\ell}{4}, x_e + \frac{\ell}{4}] \times [y_e - \frac{\ell}{4}, y_e + \frac{\ell}{4}]$ in $B(X_e, d_t/2, \lambda_e)$.
2. There are two TB occupied crossings of secondary nodes, one crossing in the rectangle $[x_e + \frac{\ell}{4}, x_e + \frac{3\ell}{4}] \times [y_e - \frac{\ell}{4}, y_e + \frac{\ell}{4}]$ and the other in the box $[x_e - \frac{\ell}{4}, x_e - \frac{3\ell}{4}] \times [y_e - \frac{\ell}{4}, y_e + \frac{\ell}{4}]$ in $B(X_e, d_t/2, \lambda_e)$.
3. There is no primary node within a distance of $D_i$ from any secondary node which composes one of the three crossings.

For each edge $e \in E$ we define two (dependent) binary random variables $A_e$ and $B_e$. We set $A_e = 1$ if condition (1) holds and $A_e = 0$ otherwise. Similarly, $B_e = 1$ if $A_e = 1$ and condition (2) holds and $B_e = 0$ otherwise. The state of an edge $e$ is denoted by $C_e = A_e \cdot B_e$, where ‘1’ stands for an open edge and ‘0’ a closed one. Note that the states of the edges are dependent; however, for $\ell > 2(2D_i + d_t)$ the state of an edge only depends on the states of its six neighboring edges.

Let $(0, 0)$ be the origin vertex. Denote by $C(0)$ the set of vertices in $\mathcal{L}$ which are connected to the origin of $\mathcal{L}$ by open paths, i.e. paths that only include open edges. Denote by $|C(0)|$ the number of vertices in $C(0)$. By the discretization $\mathcal{L}$ of the continuous model, there exists an unbounded connected component in the secondary network whenever $|C(0)| = \infty$. It follows by Proposition 3.1 and Theorem 3.3 that it suffices to prove that $Pr(|C(0)| = \infty) > 0$ for some value $\ell > 0$, to prove that there is an unbounded connected component in the secondary network a.s. By the following lemma (Lemma C.1) there exists $D_i > 0$ and also $\ell > 0$ such that $Pr(|C(0)| = \infty) > 0$. This concludes proof that there exists $D_i > 0$ that fulfills Theorem 3.5.

In percolation theory the terms “open” and “closed” can be depicted by floodgates, when the floodgate is open water can flow but when it is closed the water are blocked by the floodgate and cannot pass through it. It follows that an open path is a path that water can flow through.

**Lemma C.1:** There exists $D_i > 0$ and also $\ell > 0$ such that $Pr(|C(0)| = \infty) > 0$.

**Proof:** To prove this lemma we utilize “Peierls argument” and the definition of a “dual graph” which are described in [73, pp. 16]. Let $E_e$ be the event such that the edge $e$ is closed, that is,

$$E_e(\ell, D_t) = \{A_e = 0\} \cup \{B_e = 0\}.$$  \hspace{1cm} (38)

Denote,

$$q(\ell, D_t) = Pr(E_e(\ell, D_t)) \leq Pr(A_e = 0) + Pr(B_e = 0).$$  \hspace{1cm} (39)

Let $L'$ be the dual graph of $\mathcal{L}$. Then, by “Peierls argument”

$$Pr(|C(0)| = \infty) = Pr(\exists \text{ an infinite open path in } L')$$

$$= 1 - Pr(\exists \text{ a closed circuit in } L' \text{ which contains the origin in its interior})$$  \hspace{1cm} (40)

Denote by $\rho(n)$ the number of circuits of length $n$ which contain the origin in their interior. Then, $\rho(n)$ is upper bounded by [73, pp. 15-18]

$$\rho(n) \leq 4n3^{n-2}.$$  \hspace{1cm} (41)

Let $M(n)$ be the number of closed circuits of length $n$ which contain the origin in their interior, and let $q = q(\ell, D_t)$. Then,

$$Pr(\exists \text{ a closed circuit in } L' \text{ which contains the origin in its interior})$$

$$= Pr(M(n) \geq 1 \text{ for some } n)$$

$$\leq \sum_{n=1}^{\infty} Pr(M(n) \geq 1)$$

$$\leq \sum_{n=1}^{\infty} \rho(n)q^{n/4}$$

$$= \sum_{n=1}^{\infty} 4n3^{n-2}q^{n/4}$$

$$= \frac{4q^{1/4}}{3} \sum_{n=1}^{\infty} n \left(3q^{1/4}\right)^n$$

$$= \frac{4q^{1/4}}{3(1 - 3q^{1/4})^2}.$$  \hspace{1cm} (42)

The inequality (*) follows from the fact that in every circuit of length $n$ there are at least $n/4$ independent edges.

It follows that if $q < \left(\frac{11 - 2\sqrt{16}}{27}\right)^4$ then $Pr(|C(0)| = \infty) > 0$, and by the ergodicity the cognitive model is simultaneously connected.

To conclude the proof we show that there are $\ell$ and $D_t$ that ensure that $q(\ell, D_t) < \left(\frac{11 - 2\sqrt{16}}{27}\right)^4$.

By the law of total probability,

$$q(\ell, D_t) = Pr(E_e(\ell, D_t)) \leq Pr(A_e = 0) + Pr(B_e = 0)$$

$$\leq Pr(A_e = 0) + Pr(A_e = 0) Pr(B_e = 0 | A_e = 0)$$

$$+ Pr(A_e = 1) Pr(B_e = 0 | A_e = 1)$$

$$\leq 2 Pr(A_e = 0) + Pr(B_e = 0 | A_e = 1).$$  \hspace{1cm} (43)

Since $\lambda_s > \lambda_0 \left(\frac{d_t}{\ell}\right)$, the well known results (36) and (37) yield that $Pr(A_e = 0)$ vanishes as $\ell$ tends to infinity. Therefore, for
every $\epsilon > 0$ there exists $\ell \in \mathbb{R}$ such that $\Pr(A_e = 0) < \epsilon/3$. It is left to prove that there exists $D_\ell > 0$ such that $\Pr(B_e = 0|A_e = 1) < \epsilon/3$.

Let $\alpha$ be the (random) area in which primary nodes interfere with communication of secondary nodes in the LR or one of the TB crossings. By definition it follows that $0 \leq \alpha \leq \left( \frac{\ell^2}{4} + 2D_\ell + d_\ell \right) \left( \frac{\ell^2}{4} + 2D_\ell + d_\ell \right) = S_{\text{max}}(\ell)$. Moreover, since not all the nodes that form the LR and TB crossings are essential for the existence of these crossings, it follows that

$$\Pr(B_e = 1|A_e = 1, \alpha) \geq e^{-\lambda_\ell \alpha}. \tag{44}$$

Therefore,

$$\Pr(B_e = 0|A_e = 1) = 1 - \Pr(B_e = 1|A_e = 1) \leq 1 - \int_0^{S_{\text{max}}(\ell)} f(\alpha|A_e = 1)e^{-\lambda_\ell \alpha} d\alpha. \tag{45}$$

The function $e^{-\lambda_\ell \alpha}$ is convex with respect to $\alpha$, therefore, by Jensen’s inequality:

$$\Pr(B_e = 0|A_e = 1) \leq 1 - e^{-\lambda_\ell E(\alpha|A_e = 1)}. \tag{46}$$

Next, we upper bound $E(\alpha|A_e = 1)$. Let $K$ be the (random) number of secondary nodes in the box $\left[ x_e - \frac{3\ell^2 + 2d_\ell}{4}, x_e + \frac{3\ell^2 + 2d_\ell}{4} \right] \times \left[ y_e - \frac{3\ell^2 + 2d_\ell}{4}, y_e + \frac{3\ell^2 + 2d_\ell}{4} \right]$, and denote by $z_k = (z_1, \ldots, z_k)$ the vector of the (random) positions of these $K$ secondary nodes. Also, let $I(z_k)$ be the area of the region $\bigcup_{i=1}^{K} O(z_i, D_\ell)$. Given that $K = k$, one has that $\alpha \leq I(z_k)$ since the region $\bigcup_{i=1}^{K} O(z_i, D_\ell)$ may include secondary nodes which are not part of the crossings. Also, by definition $I(z_k)$ is upper bounded by $k \pi D_\ell^2$. We proceed to bounding $E(\alpha|A_e = 1)$,

$$E(\alpha|A_e = 1) = \int_0^{S_{\text{max}}(\ell)} \alpha f(\alpha|A_e = 1) d\alpha$$

by $\Pr(K = k|A_e = 1) = \int_0^{S_{\text{max}}(\ell)} \alpha f(\alpha|A_e = 1, K = k) d\alpha$

$$= \sum_{k=1}^{\infty} \Pr(K = k|A_e = 1) \int_0^{S_{\text{max}}(\ell)} \alpha f(\alpha|A_e = 1, K = k) d\alpha \leq \sum_{k=1}^{\infty} \Pr(K = k|A_e = 1) k \pi D_\ell^2 f(\alpha|A_e = 1, K = k) d\alpha$$

$$(a) \sum_{k=1}^{\infty} \Pr(K = k|A_e = 1) \leq \frac{1}{\Pr(A_e = 1)} \sum_{k=1}^{\infty} \Pr(K = k, A_e = 1) k \pi D_\ell^2$$

$$= \frac{1}{\Pr(A_e = 1)} \sum_{k=1}^{\infty} \Pr(K = k) k \pi D_\ell^2$$

(b) $\pi D_\ell^2 \frac{\lambda_\ell}{\Pr(A_e = 1)} \left( \frac{3\ell^2 + d_\ell}{4} \right) \left( \frac{\ell^2}{4} + d_\ell \right) \right) \right)$. \tag{47}$$

where $(a)$ follows since given that $K = k$, the interference area $\alpha$ is upper bounded by $k \pi D_\ell^2$ and $(b)$ follows since $K$ is a Poisson random variable with mean $\lambda_\ell \frac{(3\ell^2 + d_\ell)(\ell^2 + d_\ell)}{4}$.

Thus,

$$\Pr(B_e = 0|A_e = 1) \leq 1 - \exp \left( -\pi \lambda_\ell D_\ell^2 \frac{\lambda_\ell}{\Pr(A_e = 1)} \left( \frac{3\ell^2 + d_\ell}{4} \right) \left( \frac{\ell^2}{4} + d_\ell \right) \right) \right). \tag{48}$$

Further, one can choose $D_\ell > 0$ such that $1 - \exp \left( -\pi \lambda_\ell D_\ell^2 \frac{\lambda_\ell}{\Pr(A_e = 1)} \left( \frac{3\ell^2 + d_\ell}{4} \right) \left( \frac{\ell^2}{4} + d_\ell \right) \right) < \epsilon/3$. Consequently, there are $\ell, D_\ell > 0$ such that $q < \frac{11 - 2\sqrt{7}}{2\pi^2}$.

Appendix D

This appendix derives the density function $f_{r, \ell}(t)$ of the random distance $t$ between two centers of disks (see the proof of Theorem 4.3).

Let $(X_1, Y_1)$ and $(X_2, Y_2)$ be two random points uniformly distributed in $[0, \ell]^2$. It follows that $X_1, X_2, Y_1, Y_2 \sim [0, \ell]^2$ are statistically independent. Define the random variables $T_x$ and $T_y$ in the following manner:

$$T_x = X_1 - X_2, \quad T_y = Y_1 - Y_2. \tag{49}$$

Further, define the random variable $T$ and the auxiliary random variable $W$

$$T = \sqrt{T_x^2 + T_y^2}, \quad W = |T_y|. \tag{50}$$

It follows that,

$$f_{|T_x|}(|T_x|) = \frac{\ell - |T_x|}{\ell^2} \cdot \mathbb{1}_{[|T_x|] \in [0, \ell]), \tag{51}$$

and similarly,

$$f_{|T_y|}(|T_y|) = \frac{\ell - |T_y|}{\ell^2} \cdot \mathbb{1}_{[|T_y|] \in [0, \ell]). \tag{52}$$

By the transformation formula

$$f_{r, \ell}(t, w) = f_{T_x}(\sqrt{t^2 - w^2}) f_{T_y}(w) \left( \frac{2\sqrt{t^2 - w^2}}{2t} \right)^{-1}$$

$$= 4t \left( \frac{\ell - \sqrt{t^2 - w^2}}{(\ell - w)} \right) \frac{\sqrt{t^2 - w^2}}{2t} \mathbb{1}_{[w \in [0, \ell])}. \tag{53}$$

By the law of total probability

$$f_{r, \ell}(t) = \int_{\ell}^{\infty} \left( \frac{\ell - \sqrt{t^2 - w^2}}{(\ell - w)} \right) \frac{\sqrt{t^2 - w^2}}{2t} \mathbb{1}_{[w \in [0, \ell])} dw$$

$$= \mathbb{1}_{[t \in (0, \ell])} \int_{\ell}^{\infty} \left( \frac{\ell^2}{\sqrt{t^2 - w^2}} - \frac{\ell w}{\sqrt{t^2 - w^2}} - \ell + w \right) dw$$

$$+ \mathbb{1}_{[t \in (\ell, \infty)]} \int_{\ell}^{\infty} \left( \frac{\ell^2}{\sqrt{t^2 - w^2}} - \frac{\ell w}{\sqrt{t^2 - w^2}} - \ell + w \right) dw$$
\[
\begin{align*}
&= 1 \{ t \in (0, t_0) \} \frac{4t^2}{\pi^2} \left( \ell^2 \arcsin \left( \frac{w}{t} \right) \right) + \frac{\ell^2}{t^2} \sqrt{\ell^2 - w^2 - \ell w + \frac{w^2}{2}} \bigg|_{t_0}^t \\
+ &1 \{ t \in (t_0, \sqrt{t^2 - w^2}) \} \frac{4t^2}{\pi^2} \left( \ell^2 \arcsin \left( \frac{w}{t} \right) \right) + \frac{\ell^2}{t^2} \sqrt{\ell^2 - w^2 - \ell w + \frac{w^2}{2}} \bigg|_{\sqrt{t^2 - w^2}}^t \\
+ &1 \{ t \in (\sqrt{t^2 - w^2}, t) \} \frac{4t^2}{\pi^2} \left( \ell^2 \arcsin \left( \frac{w}{t} \right) \right) + \frac{\ell^2}{t^2} \sqrt{\ell^2 - w^2 - \ell w + \frac{w^2}{2}} \bigg|_{\sqrt{t^2 - w^2}}^t \\
+ &1 \{ t \in (t, \sqrt{t^2 - w^2}) \} \frac{\pi \ell^2}{2} - 2\ell t + \frac{\ell^2}{2} \\
+ &1 \{ t \in (\sqrt{t^2 - w^2}, t) \} \frac{\pi \ell^2}{2} - 2\ell t + \frac{\ell^2}{2} \\
+ &1 \{ t \in (t, \sqrt{t^2 - w^2}) \} \frac{\pi \ell^2}{2} - 2\ell t + \frac{\ell^2}{2}
\end{align*}
\]

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