MIMO DETECTION BASED ON AVERAGING GAUSSIAN PROJECTIONS

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ABSTRACT

We propose a new detection algorithm for MIMO communication systems employing a two-dimensional marginal of the Gaussian approximation of the exact discrete distribution of the transmitted data given the received data. From the 2D distributions we derive one-dimensional marginals by averaging all the 2D joint distributions related to a single input symbol. We prove that this strategy to obtain a 1D distribution from a set of not necessarily consistent 2D distributions is optimal (for a specified criterion). The improved performance of the proposed algorithm is demonstrated on several instances of the problem of MIMO detection.

***Index Terms*—** Integer Least Squares, Bayesian decoding, MIMO communication systems.

1. INTRODUCTION

We consider a MIMO communication system with n transmit antennas and m receive antennas. The tap gain from transmit antenna j to receive antenna i is denoted by $\mathbf{H}_{ij}$. In each use of the MIMO channel a vector $x = (x_1, ..., x_n)^\top$ is independently selected from a finite set of complex numbers $\mathcal{A}$ according to the data to be transmitted, so that $x \in \mathcal{A}^n$. We further assume that in each use of the MIMO channel, x is uniformly sampled from $\mathcal{A}^n$. The received vector y is given by

$$y = \mathbf{H}x + \epsilon. \qquad (1)$$

Here, noise is modeled by the random vector ϵ which is independent of x and whose components are assumed to be i.i.d. according to a complex Gaussian distribution with mean zero and with known variance σ^2. The $m \times n$ matrix $\mathbf{H}$ comprises i.i.d. elements drawn from a complex normal distribution of unit variance. The MIMO detection problem consists of finding the unknown transmitted vector x given $\mathbf{H}$ and y.

A simple sub-optimal solution, known as the Zero-Forcing (ZF) algorithm, is based on a linear decision that ignores the finite-set constraint and then, neglecting the correlation between the symbols, finding the closest point in $\mathcal{A}$ for each symbol independently. This scheme performs poorly due to its inability to handle ill-conditioned realizations of matrix $\mathbf{H}$. Somewhat better performance can be obtained by using a minimum mean square error (MMSE) Bayesian estimation for the continuous linear system. Further improvement can be achieved by the MMSE with Successive Interference Cancelation (MMSE-SIC) algorithm which is based on sequential decoding with optimal ordering [1].

Many alternative methods have been proposed to approach the ML detection performance (e.g. [2],[3]). The sphere decoding (SD) algorithm finds the exact ML solution by searching for the nearest lattice point [4]. Although SD reduces computational complexity compared to the exhaustive search of the ML solution, it is not feasible for high-order QAM and/or low SNRs. A preprocessing step based on lattice reduction (LR) has been proposed in order to enhance the performance of low-complexity suboptimal detectors and decrease time complexity of tree-search sphere decoding [5]. The performance gap of ML detection and LR based linear decoders increases greatly for a large number of antennas.

In this paper we propose a detection algorithm for MIMO communication systems employing a two-dimensional marginal of the Gaussian approximation of the exact discrete distribution of the transmitted data given the received data. From the 2D distributions we derive a one-dimensional marginals by averaging all the 2D joint distributions related to a single input symbol. We prove that this strategy to extract 1D information from several 2D distributions is optimal (for a specified criterion). Although our approach is based on partial marginalization, it is completely different from previously suggested partial marginalization methods (e.g. [6]) that are based on dividing the unknown transmitted symbols into two subsets. An outer loop is performed on all the possible values of one subset and, for each passible value, an inner optimization operation finds (Gaussian approximated) optimal values for the second subset. In our approach all symbols are treated equally and there is no division of the symbols into two subsets. We first perform Gaussian marginalization for each pair of symbols. Then we continue with a discrete marginalization to obtain a one dimensional soft decision for each transmitted symbol.

2. AVERAGING GAUSSIAN PROJECTIONS

Given the constrained linear system $y = \mathbf{H}x + \epsilon$, and a uniform prior distribution on x over a finite set of points $\mathcal{A}^n$, the posterior probability function of the discrete random vector x given y is:

$$p(x|y) \propto \exp\left(-\frac{1}{2\sigma^2}\|\mathbf{H}x - y\|^2\right), \qquad x \in \mathcal{A}^n. \tag{2}$$

The notation $\propto$ stands for equality up to a normalization constant. It can be easily verified that (2) can be written as follows:

$$p(x|y) \propto \exp\left(-\frac{1}{2}(x-z)^\top \mathbf{C}^{-1}(x-z)\right), \quad x \in \mathcal{A}^n \tag{3}$$

where $z = (\mathbf{H}^\top\mathbf{H})^{-1}\mathbf{H}^\top y$ is the least-squares estimator and $\mathbf{C} = \sigma^2(\mathbf{H}^\top\mathbf{H})^{-1}$ is its variance.

Replacing the discrete constraint $x \in \mathcal{A}^n$ in (3) with the assumption that the distribution of the transmitted vector x given y is Gaussian we obtain:

$$f_{ZF}(x|y) \propto \exp\left(-\frac{1}{2}(x-z)^\top \mathbf{C}^{-1}(x-z)\right), \quad x \in \mathcal{R}^n. \tag{4}$$

The Gaussian marginal of the multivariate Gaussian density $f_{ZF}(x|y)$ is:

$$f_{ZF}(x_i|y) \propto \exp\left(-\frac{(x_i - z_i)^2}{2c_{i,i}}\right), \qquad x_i \in \mathcal{R} \tag{5}$$

where $c_{i,i}$ is the i,i entry of the matrix $\mathbf{C}$. Going back from the Gaussian density (5) to discrete distribution, we can extract soft decision results:

$$p_{ZF}(x_i = a|y) \propto \exp\left(-\frac{(a - z_i)^2}{2c_{i,i}}\right), \qquad a \in \mathcal{A}. \tag{6}$$

Taking the most likely symbol we obtain the Zero-Forcing (ZF) approximate solution:

$$\hat{x}_i = \arg\max_{a\in\mathcal{A}} p_{ZF}(a) = \arg\min_{a\in\mathcal{A}} |z_i - a|. \tag{7}$$

Motivated by the 1D Gaussian marginalization (5) of (4) that yields the ZF method, we consider the following 2D Gaussian marginalization of (4):

$$f(x_i,x_j) \propto \exp(-\frac{1}{2}(x_i - z_i, x_j - z_j)\mathbf{D}_{i,j}^{-1}(x_i - z_i, x_j - z_j)^\top) \tag{8}$$

such that $z = (\mathbf{H}^\top\mathbf{H})^{-1}\mathbf{H}^\top y$ and $\mathbf{D}_{i,j}$ is the following 2×2 sub-matrix of the covariance matrix $\mathbf{C} = \sigma^2(\mathbf{H}^\top\mathbf{H})^{-1}$:

$$\mathbf{D}_{i,j} = \begin{pmatrix} \mathbf{C}_{i,i} & \mathbf{C}_{i,j} \\ \mathbf{C}_{j,i} & \mathbf{C}_{j,j} \end{pmatrix}.$$

Returning from the Gaussian density (8) to a discrete distribution, we obtain the following discrete joint distribution:

$$p_{i,j}(x_i = a, x_j = b|y) \propto \tag{9}$$

$$\exp\left(-\frac{1}{2}(a - z_i, b - z_j)\mathbf{D}_{i,j}^{-1}(a - z_i, b - z_j)^\top\right), \qquad a,b \in \mathcal{A}.$$

Taking the x_i marginal distribution of $p_{i,j}(x_i,x_j)$:

$$p_{i,j}(x_i = a|y) = \sum_{b\in\mathcal{A}} p_{i,j}(x_i = a, x_j = b), \quad a \in \mathcal{A} \tag{10}$$

we obtain an estimation of the true marginal probability $p(x_i|y)$ and we can obtain a hard-decision detection by taking the most likely symbol. The problem is that for $j \neq k$ the 1D discrete distributions $p_{i,j}(x_i = a)$ and $p_{i,k}(x_i = a)$ extracted from the joint distributions $p_{i,j}(a,b)$ and $p_{i,k}(a,b)$ respectively, are not necessarily equal. Unlike the 1D ZF method where for each i we obtain a single discrete distribution estimation $p_{ZF}(x_i = a)$ (6), in the 2D case for each i we obtain n–1 non-consistent joint distributions $\{p_{i,j}(x_i,x_j)|j \neq i\}$ (9) that yield n–1 different 1D distributions $\{p_{i,j}(x_i)|j \neq i\}$ (10). Thus we obtain $n-1$ different estimations of the marginal distribution $p(x_i|y)$ we want to compute.

An intuitive way to integrates all the $n-1$ estimators $\{p_{i,j}(x_i)\}$ of $p(x_i|y)$ into a single estimator is to form the unweighed average of these $n-1$ distributions, i.e.

$$\hat{p}(x_i = a|y) = \frac{1}{n-1}\sum_{j\neq i} p_{i,j}(x_i = a), \quad a \in \mathcal{A} \tag{11}$$

We show next that this distribution averaging method is optimal (in a sense defined below). The main idea is to slightly perturb the n-over-2 joint distributions $\{p_{i,j}(x_i,x_j)|i<j\}$ on $\mathcal{A}\times\mathcal{A}$ into a set $\{q_{i,j}(x_i,x_j)|i<j\}$ of joint distributions $\mathcal{A}\times\mathcal{A}$ that is consistent in the sense that the induced marginal distributions coincide, i.e.

$$\sum_{b\in\mathcal{A}} q_{i,j}(a,b) = \sum_{b\in\mathcal{A}} q_{i,k}(a,b) \tag{12}$$

for all i,j,k and for all $a \in \mathcal{A}$. This can be stated as a formal optimization problem.

Theorem 1: Let $\{p_{i,j}(a,b)|1 \le i < j \le n\}$ be a set of joint distributions on $\mathcal{A}\times\mathcal{A}$. Consider the following convex optimization problem:

$$\min_{Q} \sum_{i<j} ||p_{i,j} - q_{i,j}||^2 \tag{13}$$

such that the minimization is done over all the $Q = \{q_{i,j}|1 \le i < j \le n\} \cup \{q_i|1 \le i \le n\}$ that satisfy:

$$q_{i,j}\mathbf{1} = q_i \qquad 1 \le i < j \le n \tag{14}$$

$$\mathbf{1}^\top q_{i,j} = q_j^\top \qquad 1 \le i < j \le n \tag{15}$$

$$\mathbf{1}^\top q_i = 1 \qquad 1 \le i \le n \tag{16}$$

$$q_i \ge 0 \qquad 1 \le i \le n. \tag{17}$$

where $\mathbf{1}$ is an all-ones vector (we view the joint distributions on $\mathcal{A}\times\mathcal{A}$ as $|\mathcal{A}|\times|\mathcal{A}|$ matrices and the marginal distributions on $\mathcal{A}$ as $|\mathcal{A}|\times 1$ column vectors).

The unique solution of this convex optimization problem satisfies:

$$q_i(a) = \frac{1}{n-1}\sum_{j\neq i}\sum_{b\in\mathcal{A}} p_{i,j}(a,b) \qquad (18)$$

$$i = 1, ..., n, \qquad a\in\mathcal{A}.$$

Proof: Instead of directly addressing the optimization problem stated in Theorem 1, we first solve a relaxed optimization problem obtained by ignoring constraint (17). Then we show that the obtained solution satisfies constraint (17) and therefore the constraint is redundant.

The Lagrangian of the constrained minimization problem (ignoring constraint (17)) is:

$$\begin{aligned} L(q,\mu,\lambda) &= \frac{1}{2}\sum_{i<j}\sum_{a,b\in\mathcal{A}}(p_{i,j}(a,b)-q_{i,j}(a,b))^2 \quad (19)\\ &+ \sum_{i<j}\sum_{a\in\mathcal{A}}\mu^a_{i,j}(q_i(a)-\sum_{b\in\mathcal{A}}q_{i,j}(a,b))\\ &+ \sum_{i<j}\sum_{b\in\mathcal{A}}\mu^b_{j,i}(q_j(b)-\sum_{a\in\mathcal{A}}q_{i,j}(a,b))\\ &+ \sum_i\lambda_i(1-\sum_{a\in\mathcal{A}}q_i(a)) \end{aligned}$$

Setting the derivatives of the Lagrangian to zero we obtain:

$$\frac{\partial L}{\partial q_{i,j}(a,b)} = p_{i,j}(a,b)-q_{i,j}(a,b)-\mu^a_{i,j}-\mu^b_{j,i}=0 \qquad (20)$$

$$\frac{\partial L}{\partial q_i(a)} = \sum_{j\neq i}\mu^a_{i,j}-\lambda_i = 0 \qquad (21)$$

Summing Eq. (20) over all possible values of a and b we obtain:

$$\frac{1}{|\mathcal{A}|}\sum_{a,b\in\mathcal{A}}\frac{\partial L}{\partial q_{i,j}(a,b)} = \bar{\mu}_{i,j}+\bar{\mu}_{j,i} = 0 \qquad (22)$$

where for every $1\le i<j\le n$ we use the notation:

$$\bar{\mu}_{i,j} = \sum_{a\in\mathcal{A}}\mu^a_{i,j}, \qquad \bar{\mu}_{j,i} = \sum_{b\in\mathcal{A}}\mu^b_{j,i}.$$

Substituting Eq. (22) and constraint (14) in Eq. (20) yields:

$$\sum_{b\in\mathcal{A}}\frac{\partial L}{\partial q_{i,j}(a,b)} = \sum_{b\in\mathcal{A}}p_{i,j}(a,b)-q_i(a)-|\mathcal{A}|\mu^a_{i,j}+\bar{\mu}_{i,j}=0 \qquad (23)$$

Eq. (21) yields that:

$$\sum_{j\neq i}\bar{\mu}_{i,j} = \sum_{a\in\mathcal{A}}\sum_{j\neq i}\mu^a_{i,j} = \sum_{a\in\mathcal{A}}\lambda_i = |\mathcal{A}|\lambda_i \qquad (24)$$

Combining Eq. (23), (21) and (24) we obtain:

$$\frac{1}{n-1}\sum_{j\neq i}\sum_{b\in\mathcal{A}}\frac{\partial L}{\partial q_{i,j}(a,b)} = \qquad (25)$$

$$\frac{1}{n-1}\sum_{j\neq i}\sum_{b\in\mathcal{A}}p_{i,j}(a,b)-q_i(a)-|\mathcal{A}|\lambda_i+|\mathcal{A}|\lambda_i=0$$

Therefore, we obtain that the optimal solution satisfies:

$$q_i(a) = \frac{1}{n-1}\sum_{j\neq i}\sum_{b\in\mathcal{A}}p_{i,j}(a,b). \qquad (26)$$

So far we have ignored constraint (17). However, the optimal solution (26) is clearly non-negative since it is an average of marginal distributions that consist of non-negative numbers. Hence, constraint (17) is redundant; thus this is the optimal solution to the optimization problem posed in Theorem 1. □

Theorem 1 can be utilized to obtain the optimal strategy for extracting marginal distributions from the non-consistent n-over-2 2D joint distributions $\{p_{i,j}\}$ obtained by the 2D Gaussian marginalization. For each input symbol x_i we combine all the $n-1$ 2D discrete distributions $\{p_{i,j}|j\neq i\}$ (9) by simply averaging the $n-1$ induced marginals (18). Finally, we obtain hard decision decoding by taking the most probable symbol. Thus we obtain a 2D variant of the Zero-Forcing MIMO decoding algorithm. We dub this method the "Two-Dimensional Zero Forcing (ZF-2D) Algorithm". The ZF-2D

Input: A constrained linear LS problem: $\mathbf{H}x+\epsilon=y$, a noise level σ^2 and a finite symbol set $\mathcal{A}$.

Goal: Find (approx. to) $\arg\min_{x\in\mathcal{A}^n}\|\mathbf{H}x-y\|^2$.

Preprocessing:

Compute: $z=(\mathbf{H}^\top\mathbf{H})^{-1}\mathbf{H}^\top y$, $\mathbf{C}=\sigma^2(\mathbf{H}^\top\mathbf{H})^{-1}$.

Denote the 2×2 ij sub-matrix of matrix $\mathbf{C}$ by

$$\mathbf{D}_{i,j} = \begin{pmatrix}\mathbf{C}_{i,i} & \mathbf{C}_{i,j}\\ \mathbf{C}_{j,i} & \mathbf{C}_{j,j}\end{pmatrix}.$$

End

For each pair $1\le i<j\le n$

For every $a,b\in\mathcal{A}$, compute:

$\phi_{i,j}(a,b) = -\frac{1}{2}(z_i-a,z_j-b)\mathbf{D}^{-1}_{i,j}(z_i-a,z_j-b)^\top$

$p_{i,j}(a,b) = \frac{\exp(\phi_{i,j}(a,b))}{\sum_{c,d\in\mathcal{A}}\exp(\phi_{i,j}(c,d))}$

End

End

For each $1\le i\le n$

$q_i(a) = \frac{1}{n-1}\sum_{j\neq i}\sum_{b\in\mathcal{A}}p_{i,j}(a,b), \quad a\in\mathcal{A}$

$\hat{x}_i = \arg\max_{a\in\mathcal{A}} q_i(a)$

End

Fig. 1. The ZF-2D Algorithm.

is summarized in Fig. 1. In a similar way we can obtain a 2D version of MMSE. A 2D version of the MMSE-SIC, denoted by MMSE-SIC-2D, is formed by sequential decoding the input symbols using MMSE-2D with optimal ordering [1].

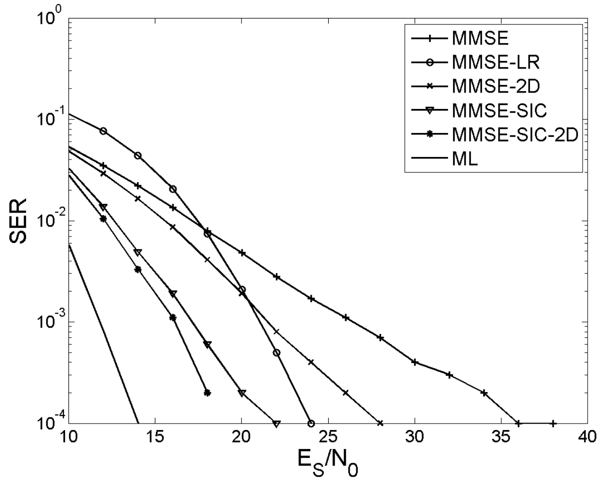

Fig. 2. Results for a 10×10, QPSK, MIMO system.

We next compute the computational complexity of the MMSE-2D. The preprocessing step of computing the covariance matrix $\sigma^2(\mathbf{H}^\top\mathbf{H} + \frac{\sigma^2}{e}\mathbf{I})^{-1}$ is common to MMSE and MMSE-2D algorithms. The complexity of this preprocessing step is $O(n^2 m)$ (if $m < n$ we can utilize the matrix inversion lemma for the matrix inversion). The complexity of computing the discrete 2D distributions and computing the average marginals is $O(n^2|\mathcal{A}|^2)$ (regardless of the number of receive antennas). The exponent operation used to obtain the discrete 2D distribution can be efficiently implemented using a lookup table.

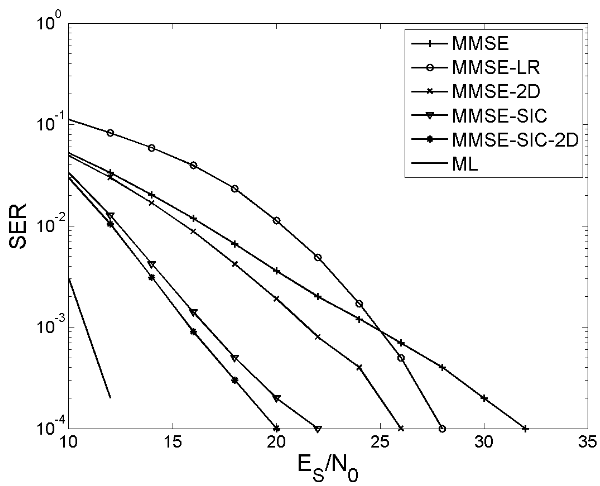

Fig. 3. Results for a 18×18, QPSK, MIMO system.

3. EXPERIMENTAL RESULTS

In this section we provide simulation results for the proposed detector over various MIMO systems. We assume a frame length of 100, i.e., the channel matrix $\mathbf{H}$ is constant for 100 channel uses. The channel matrix comprised iid elements drawn from a zero-mean normal distribution of unit variance. We used 10,000 realizations of the channel matrix. This resulted in 10^6 vector messages. The performance of the proposed algorithm is shown as a function of the variance of the additive noise σ^2. The signal-to-noise ratio (SNR) is defined as $10\log_{10}(E_s/N_0)$ where $E_s/N_0 = \frac{ne}{\sigma^2}$ (n is the number of variables, $e = \frac{1}{|\mathcal{A}|}\sum_{a\in\mathcal{A}} a^2$ and σ^2 is the variance of the Gaussian additive noise). We show here the performance of the proposed approaches based on 2D Gaussian projections. The MMSE-2D and MMSE-SIC-2D are compared to the corresponding 1D projection methods MMSE and MMSE-SIC. In the implementations of MMSE-SIC and MMSE-SIC-2D we used the same optimal ordering for the successive interference cancelation [1]. The proposed method are also compared to a MMSE decoder combined with lattice reduction (MMSE-LR)[5, 7, 8] based on the Lenstra-Lenstra-Lovasz (LLL) algorithm [9]. Finally, the MMSE-2D is compared to maximum-likelihood (ML) detection. The ML score was implemented using the Schnorr-Euchner variant of sphere decoding (SD-SE) with an infinite radius [10, 4].

Fig. 2 shows the symbol error rate (SER) versus SNR for a 10×10 complex Quadrature Phase Shift Keying (QPSK) MIMO system. Fig. 3 shows similar results for larger system size. It can be be seen from the figures that in all cases the performance of the MMSE-2D algorithm is better than the MMSE and the performance of the MMSE-SIC-2D algorithm is better than the performance of the MMSE-SIC. In all cases the improvement is significant. We can also see from the simulation figures that the MMSE-2D is always better that MMSE-LR is low SNR and as the number of the transmitted antennas increases, MMSE-2D has better performance than MMSE-LR also in high SNR.

In this paper we concentrated on hard decision decoding. The method we presented computed posterior probabilities for each transmitted symbol. Hence we can easily modify our solution to provide soft decision information needed in coded systems. The proposed method can be combined into communication systems with coding and interleaving. It is useful both for single carrier and OFDM systems. It can serve as a MIMO decoder for wireless communication systems. Using the a-posteriori probability distribution of the symbols we can easily estimate the a-posteriori probability and the likelihood ratio for the bits.

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