The Impact of MAC Protocols and Smart Antennas on the Performance of Random Ad-hoc Networks

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List of Publications

Journal Publications

A revised version of the following publications make up the main body of this dissertation:


The following publications contain some aspects of the work presented in the dissertation, as well as some related results which were not included in the final version:


Statement of Contribution

The following dissertation is a compilation of a revised version of four manuscripts. Three of the manuscripts were published in IEEE Transactions on Wireless Communications and the fourth manuscript is in the review process. The manuscripts were submitted and published as original transaction papers. IEEE Transactions on Wireless Communications is an internationally renowned journal, that publishes high-quality technical manuscripts on advances in the state-of-the-art of wireless communications.

The four manuscripts were written by the doctoral student, Yaniv George, as part of his doctoral research. The contribution of the doctoral student, Yaniv George, to the manuscripts was the most significant.

Prof. Ephi Zehavi
Dr. Itsik Bergel
Abbreviations

ASE  Area spectral efficiency
BP   Backoff probability
CSI  Channel state information
ERD  Ergodic rate density
IR-HARQ  Incremental redundancy hybrid automatic repeat request
MAC  Medium access control
MIMO  Multiple in multiple out
ORD  Outage rate density
PPP  Poisson point process
RV   Random variable
SINR  Signal to interference and noise ratio
SNR  Signal to noise ratio
TC   Transmission capacity
WANET  Wireless ad-hoc network
ABSTRACT

The impact of MAC protocols and smart antennas on the performance of random ad-hoc networks

By Yaniv George

This thesis was supervised by Professor Ephi Zehavi and Doctor Itsik Bergel

Wireless Ad-hoc Networks (WANETs) that do not depend on any infrastructure or centralized administration are primarily characterized by multi-hop communication. Their decentralized nature, robustness and flexibility makes them suitable for many applications, including vehicular, battlefield, crisis management, sensors and monitoring, tactical networks and others.

WANETs have attracted much attention from both industry and academia. However, unlike point-to-point communication, the impact of the system parameters on the performance of WANETs has remained an open issue. This dissertation provides a cross layer analysis of WANETs that utilize smart access mechanisms and smart antennas. This analysis enables a better understanding of how the wireless environment, the networks’ resources and the networks’ architecture affect the performance of WANETs. The dissertation is made up of three manuscripts that were published in IEEE Transactions, and one additional manuscript which has been submitted for publication.

The following research is based on tools from the mathematical field of stochastic geometry. To be more exact, the physical position of the nodes is modeled as a homogenous Poisson Point Process (PPP). The PPP model serves to characterize WANET performance as a function of a single parameter: user density. The results of this analysis are both simpler and more robust than the results of deterministic analysis, which depends on the specific location of each user.

The first manuscript analyzes the performance of slotted Carrier Sensing Multiple Access
(CSMA) WANETs with directional antennas. The analysis presents simple and general expressions which describe the Area Spectral Efficiency (ASE) of the network using an outage communication model. ASE expressions provide high accuracy evaluations of the optimal system parameters, such as the optimal network density, the interference threshold and the users’ rates. The second part of this work focuses on optimized networks (i.e., networks that use the optimal parameters mentioned above). For such networks we were able to derive several interesting insights. One example is that the effect of directional antennas can be summarized by a multiplicative scaling factor, which depends on the antennas’ pattern and the channel path loss. This result makes it possible to study and optimize antenna designs, and leads to a simple adaptation of many known results to the directional antennas scenario. Furthermore, we show that the ASE of CSMA WANET can be well approximated by the ASE of an ALOHA WANET multiplied by the exponent of the back-off probability. Another interesting insight is that in the low Signal-to-Noise-Ratio (SNR) regime the ASE of CSMA WANETs scales only sublinearly with the SNR. This result is better than the point-to-point case in which the capacity is proportional to the SNR. The improved scaling results from the fact that the optimal users’ density increases as the SNR decreases, and hence partially compensates for the decrease in the rate.

The next two manuscripts analyze the Ergodic Rate Density (ERD) of random ALOHA WANETs. The ERD is defined as the average mutual information between the transmitters and their intended receivers given the interference power. We present novel bounds on the ERD of ALOHA WANETs when the nodes’ position follows a PPP distribution. The lower and upper bounds are defined for general transmission and reception strategies, and an additional lower bound which is specific to receivers applying spatial interference cancellation. The gap between the general lower and upper bounds is bounded and their maximal gap is presented as a closed form expression which is only a function of the path-loss factor. The second lower bound is shown to converge to the ERD when the number of antennas grows asymptotically. The usefulness of these bounds is demonstrated in single and multiple antenna applications. For the single antenna case we consider the fixed transmission power, threshold scheduling and inverse channel strategies. For multiple
antenna WANETs we consider transmit beamforming, with or without interference cancellation and spatial multiplexing. For most applications the bounds allow simple derivation of the optimal system parameters and a closed form expression for the maximal value of the bound. For multiple antenna WANETs, the bounds also enable the evaluation of the asymptotic scaling of the ERD with the number of antennas.

In the last manuscript we study the impact of protocol synchronization on the performance of CSMA WANETs. We analyze the performance of both slotted and unslotted CSMA WANETs in the small back-off probability regime. Our main result is the derivation of simple expressions which describe the ERD of CSMA WANETs as a function of the back-off probability, the path loss exponent and the ERD of the same WANET when applying the ALOHA protocol. The ERD expressions for both the slotted and the unslotted variants are shown to grow with the back-off probability. For the slotted variant the gain of CSMA over ALOHA is equal to the back-off probability. On the other hand, the unslotted variant has lower performance, and its gain over ALOHA is a constant fraction of the back-off probability. This fraction is proved to be in the range of 0.57 to 0.67 for all cases of practical interest.
Chapter 1

Introduction

The following dissertation is a compilation of a revised version of four manuscripts. While each manuscript contains its own introduction, in this first chapter we introduce the reader to the general research subject, and describe the relationship between manuscripts. In the following we present a short theoretical background, explain why the dissertation is presented as a compilation of manuscripts, and summarize the novelty and the contribution of the dissertation.

1.1 Theoretical Background

Wireless ad hoc networks (WANETs) are wireless networks that do not depend on any infrastructure. These networks offer simplicity and flexibility and are primarily characterized by multi-hop communication. WANETs were found to be suitable for a variety of applications and therefore have attracted a great deal of attention in recent years. A significant part of the ongoing research regarding WANETs is dedicated to its performance evaluation and optimization.

Some works have evaluated WANET performance using specific network structures (e.g., [1]), but many insights come from the analysis of random networks. The seminal work of Gupta and Kumar [2] showed that the capacity of uniformly spread $n$ users scales as $O(\sqrt{n})$. Later works considered various statistical spatial models for the nodes’ location of large scale random WANETs. The use of stochastic spatial models for the analysis of wireless networks enables the utilization of known techniques and results from the field of stochastic geometry and related fields. In this work we focus on the homogeneous Poisson Point Process (PPP) [3], which is perhaps the most common
spatial process used to model the position of nodes in random WANETs. In this model, nodes in any non-overlapping areas are statistically independent.

The network performance of random WANETs is mostly determined by its Area Spectral Efficiency (ASE) which is defined as the average communication rate per unit area. The communication rates of the pairs are typically evaluated using an outage or an ergodic rate model. The outage rate model is based on a transmission at a predefined target rate and on retransmissions in the case that an outage event occurs. The Transmission Capacity (TC), [4], is defined as the network ASE subject to a fixed outage probability. The TC has been used extensively to investigate the impact of system parameters on the scaling laws of random WANETs [3, 5–7]. In the following, we refer to the optimal ASE in an outage rate model as the Outage Rate Density (ORD) of the network.

As an alternative to the outage rate model, various works have used an ergodic rate model to evaluate network performance. The ergodic rate is defined as the maximum mutual information between the transmitted and the received signals given the interferers’ activity, [8, 9]. In this case the ASE; i.e., the ergodic rate per unit area, is termed the Ergodic Rate Density (ERD) [10–12]. Among the schemes that achieve ERD are time diversity, frequency diversity, [12], and the incremental redundancy hybrid automatic repeat request (IR-HARQ) [13–15]. A comprehensive discussion of the achievability of the ERD metric is given in Chapter 3 and examples of analysis of WANETs under the ERD metric can be found in Chapters 3, 4 and 5.

The set of rules that determines the access procedure of the nodes to the medium is known as a (decentralized) Medium Access Control (MAC) protocol, [16, 17]. The two most popular MAC protocols for WANETs are the ALOHA and the Carrier Sensing Multiple Access (CSMA). The ALOHA protocol [18] and its variants (e.g., [19, 20]) are considered to be the simplest packet-based access mechanisms. The CSMA protocol [21] regulates the allowed mutual interference and hence has the potential to be more efficient and stable than the simple ALOHA protocol. The performance of both ALOHA and CSMA WANETs has been studied extensively (see [?, 22] and references therein) and some of their variants are integrated in leading communication standards [23–25].
CSMA WANETs were shown to outperform ALOHA WANETs [22, 26, 27] and their capacity gain was shown to increase with the rise in the network density. The main barrier to the analysis of random CSMA WANETs is the statistical model of positions of the active nodes (which are correlated due to the interference sensing between the nodes). Some works have modeled the nodes’ positions by repulsive point processes [28–30]. Other works, including the work in Chapter 2, model the distribution of the interferers as a PPP outside a guard zone protection around a probe receiver. This model was shown to approximate the received interference power with high accuracy [31].

Traditionally, MAC protocols are divided into slotted and unslotted protocols. In a slotted MAC protocol, all transmissions are restricted to the boundaries of predefined time-slots, whereas an unslotted protocol is based on asynchronous access to the medium. Slotted MAC protocols are typically more efficient than their unslotted variants [32] but they require at least local time synchronization. Time synchronization in WANETs may require substantial protocol overhead or additional hardware and in some cases even local time synchronization among nodes is not achievable. Hence, some of the leading communication standards support both slotted and unslotted variants (e.g., [24]). In Chapter 5 we evaluate the gain of a slotted CSMA protocol over an unslotted CSMA protocol and compare both variants to the performance of an ALOHA WANET.

The use of smart antennas techniques and directional antennas was shown to improve substantially the performance of ALOHA WANETs [33–35] and showed a potential gain for CSMA WANETs [36, 37]. Several works have investigated the tradeoff between spatial multiplexing and beamforming in Multiple-In-Multiple-Out (MIMO) WANETs. When the receivers perform interference cancellation of the undesired transmissions, the optimum number of streams was shown to be one [38]. On the other hand, when the interference was considered as noise, spatial multiplexing was shown to have a potential gain [39]. In particular, increasing the number of streams was shown to be effective when the interference is limited (i.e., large path loss factor, large number of antennas or small density of users). In the following we evaluate the performance of WANETs with smart antennas and investigate the tradeoff between different MIMO techniques. Chapter
2 derives the potential gain from the utilization of directional antennas in CSMA WANETs and Chapters 3 and 4 evaluate the gain from the utilization of beamforming, interference cancellation and spatial-multiplexing for ALOHA WANETs.

1.2 The dissertation as a compilation of manuscripts

The dissertation is presented as a compilation of several manuscripts, three of which were published in IEEE transactions, and the fourth has been submitted for publication. All manuscripts share the same framework which includes an infinite network model with randomly distributed transmitters on the plane, a single hop analysis given a predefined routing decision, a radio propagation model that consists of a path-loss attenuation and fading, a decentralized MAC protocol and utilization of the mean communication-rate per unit area as a performance metric.

Chapter 2, titled "The Spectral Efficiency of Slotted CSMA Ad-Hoc Networks with Directional Antennas" analyzes the performance of CSMA WANETs with directional antennas. The manuscript presents two simple lower bounds that yield a good prediction of the ASE of CSMA WANETs. This performance analysis led to several novel insights on the behavior of random WANETs. However, closed form expressions were obtained only for the case where all network parameters are optimized.

In order to achieve better understanding of random WANETs, Chapter 3 studies the Ergodic Rate Density (ERD) of a random network. The ERD provides a better characterization of the performance of modern communication systems, and turns out to be simpler for analysis as well.

As a first step in the analysis of the ERD, Chapter 3, titled "The ergodic rate density of ALOHA wireless ad-hoc networks" considers the ERD of a random ALOHA WANET with a homogenous PPP node distribution. This manuscript presents two novel lower bounds on the ERD, one for general transmission and reception strategies and the other for receivers with a spatial interference cancellation capability. The usefulness of the two lower bounds is demonstrated by five applications, considering both the single and the multiple antenna cases.

The lower bound for the case of interference cancellation is proven to be tight for a large
enough number of antennas. On the other hand, the tightness of the more general lower bound, introduced in Chapter 3, was studied only numerically in this manuscript. Chapter 4 manages to close this analytical gap by presenting an upper bound for the same scenario.

Chapter 4, titled "Upper bound on the ergodic rate density of ALOHA wireless ad-hoc networks" presents a novel upper bound on the ERD of random WANETs. The upper bound applies to the same model as the general lower bound presented in Chapter 3, and can support the same transmission/reception schemes and fading distributions. The formula for the upper bound is shown to be similar to the formula for the lower bound and this similarity was utilized for the quantification of the maximum gap between the bounds. Moreover, Chapter 4 presents novel expressions which serve to find the optimal number of spatial streams in MIMO WANETs.

To conclude this work, the simplicity of the ERD expressions of the ALOHA WANET was further used to acquire a better understanding of CSMA WANETs. Chapter 5 utilizes the main results of Chapter 3 in order to acquire novel insights on the gain of CSMA over ALOHA, and to shed light on the effect of synchronization on the performance of CSMA WANETs.

Chapter 5, titled "The ergodic rate density of slotted and unslotted CSMA ad-hoc networks" analyzes the ERD of slotted and unslotted CSMA WANETs in the small backoff-probability regime. The analysis presents novel lower bounds on the ERD of WANETs by applying the slotted or unslotted protocol. The lower bound expressions were shown to be useful for the optimization of network parameters.

The next section summarizes the contribution of the manuscripts.

### 1.3 Novelty and Contribution of the Manuscripts

In this subsection we detail the main contributions of the dissertation. We present the contributions according to the order of the chapters in the dissertation.

Chapter 2 presents a cross layer analysis of two-phase CSMA WANETs with directional antennas. The main result of the chapter is the introduction of two novel lower bounds on the approximated model of the ASE. The first lower bound provides insights into the effect of the
directional antenna pattern and channel properties on the ASE. The second lower bound was shown to be effective for predicting the optimal system parameters such as the users’ rates, the interference threshold and the users’ density. The second part of the chapter characterizes the performance of WANETs that use these optimal parameters (denoted here as optimized WANETs).

The analysis shows that the effect of directional antennas on the ASE of optimized WANETs can be summarized by a multiplicative scaling factor, which depends solely on the antenna pattern and the path-loss exponent. For example, for the case of a sector antenna model, directional antennas with a narrow main-lobe of width $2\pi/M$ are shown to increase the ASE by a factor of $M^{\frac{4}{\alpha}}$, where $\alpha$ is the path loss exponent. The analysis also reveals the interesting behavior of optimized WANETs in the low SNR regime: the ASE is proportional to $\rho^{1-\frac{2}{\alpha}}$, where $\rho$ denotes the SNR. Given that $\alpha > 2$, this result is significantly better than the point-to-point case, in which the capacity decreases proportionally to $\rho$. The throughput improvement stems from an increase in transmitter density as the SNR decreases, which improves the power efficiency of the WANET. Last but not least, the analysis shows that the gain of the CSMA protocol over the ALOHA protocol can be well approximated by the simple expression $e^\eta$, where $\eta$ is the allowed back-off probability.

Chapter 3 describes novel lower bounds on the ERD of ALOHA WANETs. The bounds are unique in that they present closed form expressions, which are very general and can be applied to various network setups. This generality is demonstrated by the application of these bounds to five different scenarios. One of the lower bounds is for receivers that apply interference cancellation. This bound is shown to be asymptotically tight as the number of cancelled interferers grows. The second bound is very general and poses almost no constraints on the network structure. The tightness of this bound is demonstrated by simulations in Chapter 3. This tightness is also studied analytically in Chapter 4, which introduces an upper bound for the same network setup. Interestingly, the two bounds share the same form, and vary only by a shift on the $x$-axis and a shift on the $y$-axis. Thus, the combination of the bounds from Chapter 3 and Chapter 4 gives a very good characterization of the ERD of any WANET.

The bounds of Chapters 3 and 4 are used to derive simple closed form expressions for the
ERD in various WANETs (as opposed to the outage model, where the ORD has a closed form expression only for some specific setups). The obtained closed form expressions are used to reaffirm several insights for the ERD model that were derived previously for the ORD (in most cases, the novel derivation is also much simpler). The bounds are also used to derive some new insights. For example, we show in Chapter 3 that the maximal ERD grows linearly with the path-loss factor for large values of the path-loss factor, while for fixed active user density the ERD is not necessarily a monotonic function of the path-loss factor. Furthermore, in Chapter 4 we investigate the performance of MIMO WANETs and show that if the number of streams is optimized as a function of the users’ density, spatial multiplexing can achieve a linear scaling of the ERD with the number of antennas. This result differs from previous works which required an interference cancellation scheme to achieve linear scaling (and hence required additional CSI information on the channels from participating neighbors).

Other insights that were reaffirmed using these bounds include the conclusion that a scheme with fixed transmission power is superior to an inverse-channel transmission scheme (which tries to maintain a fixed reception power), showing that the scaling of the ERD with the number of antennas is given by $N^{2/\alpha}$ for the case of beamforming and eigen-beamforming, and $N$ for the case of beamforming with interference cancellation (where $N$ is the number of antennas), demonstrating that for spatial multiplexing, increasing the number of streams does not necessarily increase the WANET’s performance.

Chapter 5 introduces novel lower bounds on the ERD of slotted and unslotted CSMA WANETs in the small back-off probability regime. In this chapter we quantify the benefits of using a synchronized protocol and present simple expressions of the expected gain of CSMA WANETs over ALOHA WANETs. The ERD of both CSMA variants is shown to monotonically increase with the allowed back-off probability. However, the unslotted CSMA variant is shown to be inferior to the slotted variant, and requires a higher back-off probability to achieve the same ERD. We also characterized this back-off gap through a closed form analytic expression. This gap was shown to be in the range of 50%-100%, and to depend solely on the path-loss factor. In this chapter we
also present the optimal node density and the optimal sensing power threshold. Interestingly, the optimal active user density of both variants of CSMA WANETs is shown to be identical to the optimal density of ALOHA WANETs.

In the four chapters that follow, we present a revised version of the four manuscripts that comprise this dissertation.
Chapter 2

The Spectral Efficiency of Slotted CSMA Ad-Hoc Networks with Directional Antennas

The Spectral Efficiency of Slotted CSMA Ad-Hoc Networks with Directional Antennas

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Abstract—The performance of wireless ad-hoc networks (WANET) is mainly limited by its self-interference. This interference can be mitigated by applying smart access mechanisms and smart antennas. In this paper we analyze the performance of WANETs applying two-phase slotted carrier sense multiple access (CSMA) mechanism and utilizing directional antennas. We present simple expressions that enable a high accuracy evaluation of the network area spectral efficiency (ASE). The results show that directional antennas only affect the ASE through a scaling factor, which depends on the antennas pattern and the channel path loss. The ASE expression also provides a simple and accurate evaluation of the optimal system parameters, including the optimal network density, interference threshold and users’ rates. In particular, the ASE gain for a CSMA WANET over an ALOHA WANET is shown to be approximated very well by the exponent of the back-off probability. The accuracy of the results and the usefulness of the optimization procedure are also illustrated through numerical simulations.

I. INTRODUCTION

Wireless ad-hoc networks (WANET) have attracted growing interest in recent years. These networks are not dependent on infrastructure such as base stations, and are primarily characterized by multi-hop communication. The decentralized nature of WANETs makes them suitable for a variety of applications, offering simplicity, scalability and flexibility.

While some interesting results on the capacity of WANETs considered specific network structures (e.g., [1]), many works had considered random networks. Gupta and Kumar [2] studied the capacity scaling law of WANETs. They showed that the capacity of a WANET with $n$ users, employing omni-directional antenna and slotted ALOHA protocol, scales as $O(\sqrt{n})$.

The performance of WANETs is typically characterized by their area spectral efficiency (ASE), which measures the average rate of successful transmissions per unit area. Weber et al., [3], defined the transmission capacity (TC): the maximum ASE subject to a maximal outage probability constraint, and presented closed-form asymptotic lower and upper bounds on the TC of WANETs employing the ALOHA protocol. In a later work together with Jindal, [4], they studied the TC of WANETs using various transmission strategies over a fading channel, and demonstrated that threshold scheduling is superior to channel inversion.

In order to increase the ASE, many WANETs employ a MAC protocol that can reduce the probability of message collisions. Carrier sensing multiple access (CSMA), [5], is one of the most popular access mechanism. The performance of CSMA WANETs has been studied extensively (e.g., [6]–[10]), and many variants were considered. In this work we focus mainly on the slotted CSMA variant, [11], where all users’ transmissions are aligned to predetermined time slots. CSMA WANETs were shown to outperform ALOHA WANETs, [12], [13], and their throughput gain was shown to increase with the rise in network density. However, in order to achieve this capacity gain, the CSMA WANET parameters (e.g., the carrier sense threshold) must be optimized. The optimal parameters were shown to depend on channel path loss and on the fading (or no fading) type [14], [15]. The CSMA mechanism uses the network resources and increases the system overhead. Several works (e.g., [16]) have considered throughput optimization while also taking into account the CSMA management overhead.

Unfortunately, detailed analysis of random CSMA WANETs is very complicated, primarily due to the carrier sensing interactions between users that result in an intractable distribution of active users. Hence, each performance analysis of random CSMA WANETs has been forced to use approximations. For example, several works, [17]–[19], used repulsive point processes to model the CSMA participating nodes. Another approach, suggested by Hasan and Andrews [20], modeled the distribution of the interferers’ location in a CSMA WANET as a homogeneous Poisson point process (PPP) with a guard zone around a probe receiver. This approximation was validated by comprehensive simulations and tests. They also used a second approximation, in which the power of the aggregate interference was modeled as Gaussian random variable. However, this second approximation was shown to be realistic solely for large spreading factors. The interference of a Matérn hard core process of type II and PPP was investigated in [21]. The gap between the mean interference of the two processes with the same density of nodes was shown to be very small (i.e., bounded by 1 dB). A preliminary publication from this current research, [22], had shown that without the Gaussian approximation, the PPP approximation can model the aggregate interference of CSMA WANET very accurately.

The use of smart antennas techniques in WANETs can reduce interference between the users, and hence leads to an increase in network capacity. [23], [24]. WANETs with smart antennas have been analyzed mostly for ALOHA networks. Utilization of directional antennas, [25]–[28], was shown to significantly improve the capacity of WANETs. Beamforming and sector (directional) antennas were shown to be superior to antenna selection or space time coding [29].

Several works analyzed WANETs utilizing CSMA protocols with directional antennas. Multi-layer design, [30], [31], showed a potential gain when utilizing directional antennas for CSMA WANET. Bounds on this gain were shown to
depend on the antenna characteristics such as the antenna beamwidth and mainlobe-to-sidelobe-ratio, [32], [33]. New medium access protocols, [34]–[37], based on CSMA and taking directional antennas into account were shown to outperform the 802.11, [38], standard protocol. Also, switched beams where shown to perform almost as good as steered beams, [39], in the case of WANETs utilizing CSMA protocol with antenna arrays.

Hunter et al. [40] studied a CSMA WANET utilizing spatial multiplexing transmission over fading channels. The authors presented an expression for the successful transmission probability of MIMO CSMA. Note that in this work, the effect of the guard zone was approximated by a PPP with a non-homogeneous density of transmitters which increases with the distance. A preliminary publication from the work presented herein analyzed the ASE gain from utilizing directional antennas in CSMA WANETs, [22]. This gain was shown to depend solely on the antennas directivity and the channel path loss.

In this paper we analyze the ASE of two-phase CSMA WANETs with directional antennas. The model assumes general antenna pattern, general pair-distance distribution and general channel-fading model. We present simple expressions that enable a high accuracy evaluation of the network ASE. These expressions provide insight into the impact of the backoff probability, channel properties and antennas directivity on the ASE. They can also be used to determine the optimal system parameters, including the interference threshold, the pair communication rates and the user density. The paper also presents numerical results that demonstrate the accuracy and the usefulness of the derived expressions.

The rest of this paper is organized as follows: Section II describes the system and approximation models, section III details the performance analysis and presents two lower bounds on the performance of slotted CSMA networks with directional antennas, section IV presents our insights from the ASE bounds and Section V covers numerical and simulation results. Concluding remarks are found in section VI.

II. SYSTEM MODEL

A. Network Model

We assume a decentralized wireless ad-hoc network utilizing a two-phase slotted CSMA medium access control (MAC) protocol. We focus on the access protocol and the physical layer processing, and adopt a probabilistic simplified model to characterize the operation of higher communication layers [19], [29].

The routing mechanism is assumed to have prior knowledge on nodes’ availability, position and orientation. On the other hand, the routing mechanism has no knowledge on the instantaneous channels fading and the instantaneous activity of the nodes in the network. Note however, that due to the use of directional antennas, a special attention is required to insure that the antennas of communicating nodes pairs will point toward each other. This is known as the deafness problem, and was studied for example in [41], [42]. In this work we do not address the deafness problem, and assume that it is solved by the routing mechanism. The exact effect of this issue on practical system performance is left for future study.

For the position of nodes we adopt the popular Poisson bipolar network model [43], [44]. In this model, each point of the Poisson pattern represents a node of the WANET (a potential transmitter) and has an infinite backlog of packets to transmit to its associated receiver. The latter is located at a fixed or random distance from its transmitter, and its orientation obeying a uniformly randomly distribution. This model is used to represent a snapshot of a multi-hop WANET: the transmitters are relay nodes and are not necessarily the sources of the transmitted packets; similarly, the receivers need not be the final destinations. The spatial density of the potential transmitters is given by \( \lambda_0 \).

The considered two-phase slotted CSMA protocol is based on periodic slots. Each slot is assembled of the following two phases: An access sub-slot (sometimes termed CAP - contention access period) and a data sub-slot (sometimes termed CFP - contention free period). During the access sub-slot, each contending pair waits a random time before transmitting its RTS message. These random times create an ordering between the contending pairs, giving higher priority to pairs with shorter wait time. When its turn comes, a transmitter transmits a request to send (RTS) message. Based on this RTS message and on previous measurements, the receiver preforms an initial estimate of the prospects of successful detection, using a threshold test on the interference from each of the winning transmitters. If the threshold test succeeds, the receiver decides whether a message can be successfully received. If so, the receiver transmits a clear to send (CTS) message. This type of protocol have been analyzed for example in [45]. It is also part of the low-rate wireless personal area networks (LR-WPANs) standard, [11] (although the protocol is not specific for low rate networks). For simplicity, the duration of the access sub-slot is assumed to be significantly larger than the duration of the RTS-CTS process. Thus, the probability of RTS-CTS messages collisions is negligible, and its effect is not taken into account in this work (see for example [10] and reference therein).

Each node is equipped with a single directional (sector) or omnidirectional antenna. The power received at receiver \( j \) from transmitter \( i \) is:

\[
W_{i,j} = \rho D_{i,j}^{-\alpha} V_{i,j} G(\theta_{R_{i,j}})G(\theta_{R_{j,i}})
\]

where \( \rho \) is the transmission power; \( D_{i,j} \), and \( V_{i,j} \) are the distance between transmitter and receiver and the additional channel gain (e.g., due to fading) respectively, \( \alpha > 2 \) is the path loss exponent and \( G(\theta) \) is the antenna gain pattern at angle \( \theta \). We assume that each contending transmitter-receiver pair point their antennas toward each other. This direction (of the desired transmitter/receiver) is considered as the zero angle for each node. Considering the relative locations of receiver \( j \) and transmitter \( i \), \( \theta_{R_{i,j}} \) denotes the relative angle in which this direction is seen by transmitter \( i \), and \( \theta_{R_{i,j}} \) denotes the relative angle in which this direction is seen by receiver \( j \) (i.e., \( \theta_{R_{i,j}} = \theta_{R_{j,i}} = 0 \) for all \( i \)). From the definition of the bipolar model we can deduce that the location of the \( i \)-
th transmitter is statistically independent of the location of the j-th receiver for $i \neq j$. Hence, the relative angles, $\theta_{Ti,j}$ and $\theta_{Ri,j}$, are statistically independent, and both are uniformly distributed in the range between 0 and $2\pi$, for any $i \neq j$ (e.g., [22], [29]). For normalization we also assume that the antenna gain pattern is normalized so that $G(0) = 1$.

In this model the fading random variable (RV), $V_{i,j}$, is independent and identically distributed (i.i.d) for all $i, j$, and it is also statistically independent of all distance and antenna gain variables. All results presented henceforth hold for any channel fading, described by the distribution of $V_{i,j}$. The channel gains are assumed to remain constant throughout a slot period.

The $k$-th contending pair becomes a winning pair if the pair and all previous winning pairs satisfy:

$$W_{ij} < \rho \delta, \forall i \neq j \in \{S_k \cup k\}$$

(2)

where $S_k$ is the set of all pairs that were termed winning pairs prior to the turn of pair $k$, and $\delta$ is the allowed interference threshold. At the end of the access sub-slot, all winning transmitters ($S = \cup S_k$) start to transmit data all through the data sub-slot (and (2) is satisfied for all $i \neq j \in S$).

Note that a winning pair gains access to the channel, but this does not guarantee the correct decoding of the data by the receiver. This is because the threshold test is based on a single interfering user, while message reception depends on the aggregate interference from all users. If (2) is not satisfied for pair $k$ we say that a contention occurred, and pair $k$ backs off. In this case the data that transmitter $k$ planned to transmit will be transmitted in a future slot in which pair $k$ will be a winning pair. Denoting by $\lambda_c$ the density of the winning pairs, the probability for a pair to back off when applying the parameters $\lambda_c$ and $\delta$ is given by:

$$P_b(\lambda_c, \delta) \triangleq 1 - \frac{\lambda_c}{\lambda_p}$$

(3)

Note that although $\lambda_c$ is actually determined by $\lambda_p$ and $\delta$ we prefer to address the reverse relation, and consider $\lambda_p$ as a function of $\lambda_c$ and $\delta$. In the following the performance of the slotted CSMA protocol is also compared to the performance of the slotted aloha protocol. To that end, it is useful to note that the model of (2) includes the slotted aloha protocol by setting $\delta \to \infty$. In that case $\lambda_c = \lambda_p$ and $P_b(\lambda_c, \delta) = 0$.

Due to our assumption on the duration of the access sub-slot, the price of a back-off event in our model is negligible. It only costs in a redundant RTS-CTS messages, which are assumed to be of negligible length. However, taking into account other practical considerations, we note that a high back-off probability (BP) has a major impact on the network. Each time a user backs off, it will need to delay for several time slots until it can try to transmit again. Hence, a high BP can result in very large delays and even network instability. Furthermore, large BP will require the network to transmit many redundant protocol messages. To avoid protocol messages collisions, the network will require a longer access sub-slots which will reduce the network efficiency. Hence, a reasonable network architecture will optimize network performance at the data sub-slots while limiting the penalty from the back-off events.

In the following, the performance of a CSMA WANET at the data sub-slots is analyzed under the constraint of a maximal allowed BP.

For illustration purposes we present Fig.1 which describes a sample realization of a small area of the network. In this figure, triangles and squares depict the locations of contending transmitters and receivers respectively, and the curve around each node represents its antenna gain pattern (sector antenna with main lobe of $3\pi/4$ in this example). The distance between each transmitter and its pair receiver in this example is assumed to be fixed. The pairs’ number represents their scheduling order during the access sub-slot (according to the random wait times used by each transmitter). In the realization illustrated in this figure, four pairs were winning (pair number 0,2,3 and 4) while pairs number 1 and 5 were disabled. Note that in this illustration pair 1 backed off due to the interference from transmitter 0 while pair 5 backed off in order to avoid interference to receiver 2.

The optimal strategy for the selection of powers and rates in WANETs over fading channels is unknown. Several works have considered the case of constant power and constant rate for all users. By contrast, in this work we consider a constant power and adaptive rate strategy. In this adaptive rate strategy each pair modifies its transmission rate according to the quality of the link, [1], and the distance between the transmitter and the desired receiver. Using this adaptive rate strategy, pair $i$ encodes its data at a rate

$$R_i = \log_2 \left(1 + \mu V_{i,i} d_i^{-\alpha} \right)$$

(4)

where $d_{i,i}$ is the distance between a transmitter-receiver of pair $i$ and $\mu$ is a rate factor, which is a design parameter that indicates the allowed interference level. Note that this strategy ensures that the outage event of the $i$-th pair is statistically independent of $V_{i,i}$ (which also results in a simpler analysis). Note that all results presented henceforth hold for any distribution of the pair distance, $d_{i,i}$.

In the following, we focus on the analysis of the achievable data rates in the data sub-slots. We use the shift invariant property of the system, [46], to analyze the performance of the network using user 0 as a probe receiver. Without loss of generality, we assume that the probe receiver is located at the origin, and that it is a part of the winning set. For notation simplicity, in the following we drop the probe receiver index.
throughout the paper. We also define the set $\mathcal{S}$ as the set of winning pairs, $\mathcal{S}$, excluding the probe pair.

The aggregate interferer, measured at the probe receiver, can be written as:

$$\rho I = \sum_{i \in \mathcal{S}} W_i = \rho \sum_{i \in \mathcal{S}} D_i^{-\alpha} G_i V_i$$

where $I$ denotes the normalized aggregate interference and $G_i \triangleq G(\theta_0)G(\theta_k)$ is the combined antenna gain. The desired signal, measured at the probe receiver, is $\rho V_0 d_0^{-\alpha}$. Without loss of generality, in the following we set the thermal noise variance to 1. Hence, the signal-to-noise-and-interference-ratio (SINR) can be written as

$$\rho I d_0^{-\alpha} (1 + \rho I)^{-1}$$

and we will refer to $\rho$ as the signal-to-noise-ratio, SNR). Using Shannon’s theory, [47], assuming a single user decoder over a Gaussian codebook and sufficiently long block length, a transmission can be decoded successfully if $R_0 < \log_2 \left(1 + \frac{\rho V_0 d_0^{-\alpha}}{1 + \rho I} \right)$. Comparing to (4), the outage probability is:

$$P_o(\lambda_\eta) = \Pr \left( \frac{\rho V_0 d_0^{-\alpha}}{1 + \rho I} < \mu V_0 d_0^{-\alpha} \right)$$

$$= 1 - \Pr \left( \lambda < \frac{1}{\mu} - \frac{1}{\rho} \right)$$

$$= 1 - \Pr \left( \lambda < \frac{1}{\beta} \right)$$

where in the following we use the parameter $\beta \triangleq \frac{\mu \rho}{\rho + \beta}$ to control the rate optimization (instead of $\mu$). In order to guarantee that no single transmission will cause an outage event, $\delta < \frac{1}{\beta}$ should be chosen. In CSMA WANETs, $\delta > \frac{1}{\beta}$ seems to be impractical. Nevertheless, the analytical treatment below benefits greater insights by allowing $\delta$ and $\beta$ to take any positive value.

A useful measure of the network performance is the average data rate per unit area, defined as:

$$R^M(\lambda_\eta) = \lambda_\eta \cdot (1 - P_o(\lambda_\eta, \delta, \beta))$$

$$R^M(\lambda_\eta) = \lambda_\eta \cdot \left(1 - \Pr(\lambda < \frac{1}{\beta})\right)$$

$$= \lambda_\eta \cdot \left(1 - \Pr(\lambda < \frac{1}{\delta})\right)$$

Note that in contrast to the back-off events, an outage event is well modeled in this work. Failed messages are transmitted (and create interference) throughout the data sub-slot, and hence have a negative effect on the network throughput. Thus, we do not limit the outage probability, and allow it to take the optimal value that will maximize the performance. Also note that $\lambda_p$ represents the joint density of new transmissions and retransmissions. Therefore, the actual density of new transmissions is $\lambda_c(1 - P_o) = \lambda_p(1 - P_h)(1 - P_0)$.

To optimize the system we define the set $C^M(\eta)$, which includes all pairs $(\lambda_\eta, \delta)$ that satisfy the BP constraint, i.e.,

$$C^M(\eta) = \{(\lambda_\eta, \delta) : P_h(\lambda_\eta, \delta) \leq \eta\}$$

where $\eta$ is the maximum allowed BP. The area spectral efficiency (ASE) is the maximum average rate given a maximal BP:

$$A(\eta) = \max_{\beta, (\lambda_\eta, \delta) \in C^M(\eta)} R^M(\lambda_\eta, \delta, \beta).$$

### B. Approximation Model

The analysis of the slotted CSMA system exhibits several difficulties that need to be removed. The prime difficulty is the distribution of the winning transmitters, which is quite complex. In the simplest case of an omnidirectional antenna and $d \to 0$, this distribution is known as a Matern type III distribution [48]. This distribution was studied extensively in many contexts (e.g., [49] and references therein), but the results are insufficient for system analysis. In this work, we adopt the approximation of the distribution of the winning transmitters as a PPP with density $\lambda_c$ outside of the guard-zone of the probe receiver [20], [50]. Denoting the interference received by the probe user in the approximating PPP by $\rho I$, the resulting approximate sum-rate is:

$$R^M(\lambda_c, \delta, \beta) \triangleq \lambda_c \cdot \Pr\left(I \leq \frac{1}{\beta}\right)$$

$$R^M(\lambda_c, \delta, \beta) \triangleq \lambda_c \cdot \Pr\left(I \leq \frac{1}{\beta}\right)$$

This approximation is not sufficient, as one still needs to characterize the relation between $\lambda_p$ and $\lambda_c$, i.e., the BP. An exact characterization of the BP is also intractable, and we need a second approximation. For this purpose we use again the PPP approximation together with a small BP approximation as described below:

Clearly, increasing the density of the contending pairs, $\lambda_p$, increases the density of the winning pairs, $\lambda_c$. The specific relationship between the increments of these two densities depends on the CSMA parameters at a given working point. In the following we prefer to express the working point of the network as a function of the power threshold and the winning pairs’ density parameters. Hence, in a given network state, an increment $d\lambda_p$ in the density of contending pairs will result in an increment $d\lambda_w = \frac{d}{\rho} \rho V_0 d_0^{-\alpha}$ in the density of winning pairs, where $\rho V_0 d_0^{-\alpha}$ is the probability of a contending pair to become a winning pair when the network already contains winning pairs at density $\lambda_w$, and using a power threshold of $\alpha$. Thus, the derivative of the density of the contending pairs with respect to the density of the winning pairs can be written as:

$$\frac{d\lambda_w}{d\lambda_c} = \frac{1}{\rho V_0 d_0^{-\alpha}}$$

According to the PPP approximation, the point process of the winning transmitters is a PPP, and the probability of a contending pair to become a winning pair will be:

$$P_w(\lambda_c, \delta) \approx \left[ e^{-S(\lambda_c, \delta)} \right]$$

where $S(\lambda_c, \delta)$ is the average number of winning pairs in contention with a new contending pair. Note that for low $\lambda_c$ the interaction between pairs is negligible, and hence the PPP approximation is very good.

The average number of contention pairs can be written as:

$$S_{RUC}(\delta) = \frac{1}{\lambda_c} \left[ \sum_{j=1}^{\infty} \Pr \left(\left[ (W_{ij} \geq \rho \delta) \bigcup (W_{ji} \geq \rho \delta) \right] \right) \right]$$

where
is the effective pair contention area. Following the PPP approximation, $S_{R,LC}(\delta)$ does not depend on $\lambda_c$. Substituting (12) and (13) into (11) and integrating leads to:

$$
\lambda_p \approx \int_0^{\lambda_c} e^{\lambda S_{R,LC}(\delta)} d\lambda = \frac{e^{\lambda_c S_{R,LC}(\delta)} - 1}{S_{R,LC}(\delta)}. \tag{15}
$$

Substituting (15) into (3) results in:

$$
P_B(\lambda_c, \delta) \approx \frac{e^{\lambda_c S_{R,LC}(\delta)} - 1 - \lambda_c S_{R,LC}(\delta)}{e^{\lambda_c S_{R,LC}(\delta)} - 1} = \sum_{j=1}^{\infty} \frac{(\lambda_c S_{R,LC}(\delta))^j}{j!} \left[ \sum_{j=1}^{\infty} \frac{(\lambda_c S_{R,LC}(\delta))^j}{j!} \right] \tag{16}
$$

where the right hand side term used the sequence $e^x = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!}$. For small enough $\lambda_c S_{R,LC}(\delta)$, Equation (16) is approximated by:

$$
P_B(\lambda_c, \delta) \approx \frac{1}{2} \lambda_c S_{R,LC}(\delta). \tag{17}
$$

As shown in the numerical section below, this first order approximation, (17), is very accurate even for quite high densities (surprisingly, it is even more accurate than (16)). Hence, we adopt (17) as a useful approximation for the BP.

Using (17) we define the allowed parameter set for the approximated model by:

$$
\tilde{C}^M(\eta) \triangleq \{(\lambda_c, \delta) : \frac{\lambda_c S_{R,LC}(\delta)}{2} \leq \eta\} \tag{18}
$$

and the ASE approximation is:

$$
\tilde{A}(\eta) \triangleq \max_{(\lambda_c, \delta) \in \tilde{C}^M(\eta)} \tilde{R}^M(\lambda_c, \delta, \beta) \tag{19}
$$

where $\eta$ is the maximum allowed BP. The accuracy of (19) is illustrated in the numerical section below where we show that both the PPP model and the first order approximation of the BP are accurate for a large range of BP values.

### III. PERFORMANCE ANALYSIS

In this section we present two lower bounds on $\tilde{A}(\eta)$. The first lower bound (Lower bound A) is simpler and can be used to gain insight into the behavior of the ASE as a function of the BP, the antenna pattern and the channel properties. The second lower bound is tighter, and is useful for performance evaluation and for the optimization of the system parameters. In order to simplify the expressions we define the auxiliary parameter $\kappa \triangleq \frac{\beta}{\alpha}$ where $\kappa \in (0, 1)$.

#### A. Lower Bound A

**Theorem 1:** A lower bound on $\tilde{A}(\eta)$ with maximal BP of $\eta$ is:

$$
\tilde{A}(\eta) \geq \left( \frac{\Phi_{M}}{\pi \Phi} \right) \cdot \max_{\zeta} Q(\zeta, \eta, \alpha) \cdot \max_{\beta} U(\beta, \alpha, \rho) \tag{20}
$$

where

$$
\Phi \triangleq E[V^n] \tag{21}
$$

$$
\Psi_{M} \triangleq (E[G^n])^{-1} \tag{22}
$$

$$
Q(\zeta, \eta, \alpha) \triangleq \zeta \left( \int_0^{\zeta} L^{-1} \{ \Xi(s) \} dt \right) \tag{23}
$$

$$
\Xi(s) \triangleq \exp(\eta(1-e^{-s\eta})) - s^\kappa \gamma(1-\kappa, s\eta^{-\frac{1}{\kappa}}) \tag{24}
$$

$$
U(\beta, \alpha, \rho) \triangleq E \left[ \beta^{-\alpha} \log_2(1 + \left(\frac{\beta \rho}{\beta + \beta}\right) V^d - \alpha) \right] \tag{25}
$$

$G$ is a R.V. which represents the combined antenna gain of the probe receiver from an interfering transmitter, and $\gamma(u, x)$ is the lower incomplete gamma function, which is the solution to $\int_0^{\infty} t^n e^{-t} dt$, and $L^{-1}$ is the inverse Laplace transform.

**Theorem 1:** The characteristics of the lower bound, and the insights gained from it on the network performance are given in Section IV below.

**Proof of Theorem 1.** We start the proof by restricting the parameter space to a smaller subspace (hence lower bounding the ASE). Define the set of parameter pairs $B^M(\eta)$:

$$
B^M(\eta) \triangleq \{(\lambda_c, \delta) : \frac{\Phi_{M}}{\pi \Phi} \lambda_c \pi \delta^{-\kappa} = \eta\} \tag{26}
$$

**Lemma 1:** Any parameter pair in the set $B^M(\eta)$ will result in an approximated BP which is smaller or equal to $\eta$, i.e.,

$$
B^M(\eta) \subset \tilde{C}^M(\eta) \tag{27}
$$

**Proof of Lemma 1.** The contention area, (14), is defined as the probability of a union of two events. We term this events CTS contention (contention due to interference from receiver 0) and RTS contention (contention due to interference to transmitter 0). Considering the CTS contention alone, following (2), its area is:

$$
S_{CTS} \triangleq \frac{1}{\lambda_c} E \left[ \sum_{j=1}^{\infty} \Pr(W_{0j} \geq \rho \delta) \mid \{D_j\} \right]
$$

$$
= \frac{1}{\lambda_c} E \left[ \sum_{j=1}^{\infty} \Pr(D_j \leq (G_j V_j \delta^{-1})^{\frac{1}{2}}) \right]
$$

$$
= \frac{1}{\lambda_c} E \left[ \lambda_c \pi (G_j V_j \delta^{-1})^\kappa \right]
$$

$$
= \frac{\Phi_{M}}{\pi \Phi} \pi \delta^{-\kappa} \tag{28}
$$

where the second equality uses (1), the third equality is achieved by calculating the average number of nodes inside a circle with radius $(G_j V_j \delta^{-1})^{\frac{1}{2}}$ for PPP with density $\lambda_c$, and the last equality results from (21) and (22). The symmetry of the CTS and RTS properties leads to $S_{R,TS} = S_{CTS}$. Using the union bound on the RTS and CTS contention events results in:

$$
S_{R,LC}(\delta) \leq 2 \frac{\Phi_{M}}{\pi \Phi} \pi \delta^{-\kappa} \tag{29}
$$

Substituting (29) in (18) and comparing to (26) leads to (27) and completes the proof of the lemma. 

\[\square\]
The maximal sum-rate that can be achieved by applying the parameter pairs in $B^M(\eta)$ is defined as:
\[
\hat{R}_M^M(\eta) \triangleq \max_{(\lambda_c, \delta) \in B^M(\eta)} \hat{R}_M^M(\lambda_c, \delta, \beta). 
\] (30)

Using (19), (30) and Lemma 1 leads to:
\[
\tilde{A}(\eta) \geq \hat{R}_M^M. 
\] (31)

Equations (1) and (2) indicate that a probe receiver disables transmitters which satisfy $D_i^{-\alpha} G_i V_i \geq \delta$. Therefore, transmitter $i$ will be disabled if its distance from the probe receiver is smaller than $(G_i V_i / \delta)^{1/\alpha}$. The characteristic function of the aggregate interference, measured at the middle of a circle with a disabled radius $A$ within a 2-dimensional PPP with density $\lambda$ and a fading variable $Y$ is (e.g., [50], [51]):
\[
\Phi(s) = \exp \left( -\lambda \int_A^\infty E\left[ 1 - e^{-s \sqrt{\pi t}} \right] 2\pi \pi dt \right). 
\] (32)

Using the superposition property of the Poisson shot noise [52], the characteristic function of the normalized aggregate interference, $\tilde{I}$, can be written as:
\[
\Phi_{\tilde{I}}(s) = \exp \left( -\lambda_c E_G V \int_0^\infty \left( 1 - e^{-s \sqrt{\pi t}} \right) 2\pi \pi dt \right) 
= \exp \left( -\lambda_c E_{[G V]} \int_0^\infty \left( 1 - e^{-s \sqrt{\pi t}} \right) 2\pi \pi dr \right) 
= \exp \left( -\frac{\lambda_c E_{[G V]} \Phi_0}{\pi} \int_0^\infty \left( 1 - e^{-s \sqrt{\pi t}} \right) 2\pi \pi dt \right) 
\] (33)

where the first line uses $Y = G V$, the second line results from the substitution $r = t \left( \frac{\delta}{G V} \right)$ and the third line uses (21), (22). For $(\lambda_c, \delta) \in B^M(\eta)$ Equation (33) can be written as:
\[
\Phi_{\tilde{I}}(s) = \exp \left( -\frac{\eta}{\pi} \int_0^\infty \left( 1 - e^{-s \theta \eta^2} \right) 2\pi \pi dt \right) 
= \exp \left( -\frac{1}{\pi} \int_0^\pi \left( 1 - e^{-s \theta \eta^2} \right) 2\pi \pi dr \right) 
\] (34)

where the second equality uses the substitution $r = \sqrt{\eta} t$. As shown in [50], $\Xi(s)$, defined in (24), can be also written as:
\[
\Xi(s) = \exp \left( -\frac{1}{\pi} \int_0^\pi \left( 1 - e^{-s \theta \eta^2} \right) 2\pi \pi dr \right) 
\] (35)

and therefore equation (34) is equivalent to $\Phi_{\tilde{I}}(s) = \Xi(s \theta \eta^2)$. Defining $\xi(t) = \mathcal{L}^{-1}\{\Xi(s)\}$ and using the Laplace scaling property we can write:
\[
\int_0^\frac{1}{\beta} \mathcal{L}^{-1}\{\Xi(s \theta \eta^2)\} dt = \frac{1}{\theta \eta^2} \int_0^\frac{1}{\beta} \xi \left( \frac{t}{\theta \eta^2} \right) dt 
= \frac{1}{\theta \eta^2} \int_0^\frac{1}{\beta \eta^2 \theta^2} \xi(t) \cdot t dt. 
\] (36)

Thus, the probability for a successful reception can be written as:
\[
\Pr \left( I \leq \frac{1}{\beta} \right) = \frac{1}{\beta} \int_0^\pi \mathcal{L}^{-1}\{\Phi_{\tilde{I}}(s)\} dt 
= \frac{1}{\beta} \int_0^\pi \mathcal{L}^{-1}\{\Xi(s)\} dt 
\] (37)

where the second equality uses the definition:
\[
\xi(t) \triangleq \eta(t \theta \eta^2). 
\] (38)

Substituting (37) into (10) and using (26) and (30) simplifies the lower bound to:
\[
\hat{R}_M^M = \frac{\psi_M}{\pi \Phi} \eta \delta \eta \left( \int_0^\pi \mathcal{L}^{-1}\{\Xi(s)\} dt \right) 
= \frac{\psi_M}{\pi \Phi} \max_Q (\zeta, \eta, \alpha) \max_{\beta} \mathcal{U}(\beta, \alpha, \rho) 
\] (39)

where the second line also uses (38). The proof of Theorem 1 is completed by joining (23), (25), (31) and (39).

B. Lower Bound B

Lower bound A can be improved by one additional step in which the maximal rate, obtained in (20), is associated to smaller BP values as follows:

Theorem 2: $\hat{A}(\eta)$ is lower bounded by:
\[
\hat{A}(\eta \cdot \Upsilon(\delta^*)) \geq \frac{\psi_M}{\pi \Phi} \max_Q (\zeta, \eta, \alpha) \max_{\beta} \mathcal{U}(\beta, \alpha, \rho) 
\]

where
\[
\beta^* = \arg \max_{\beta} \mathcal{U}(\beta, \alpha, \rho) 
\zeta^* = \arg \max_{\zeta} Q(\zeta, \eta, \alpha) 
\delta^* = \frac{\zeta^*}{\beta^*} \frac{\pi \eta \delta \eta}{2} 
\]

and
\[
\Upsilon(\delta) \triangleq \frac{S_{R, \zeta}(\delta)}{2 \pi \sqrt{\eta \delta \eta}}. 
\] (41)

This bound, termed henceforth Lower bound B, is slightly more complicated than the previous one, as one cannot evaluate it directly for a desired BP value, $\eta$. Nevertheless, it is superior to Lower bound A for any value of $\eta$ (see Lemma 2). The bound can also be used to determine the values of the network parameters (rate factor, interference threshold and contending transmitters density). These optimized parameters are given by $\beta^*$, $\delta^*$ and
\[
\lambda^*_p = \frac{\psi_M}{\pi \Phi} \eta \cdot \delta^* \cdot \frac{1}{\Upsilon(\delta^*)}. 
\] (42)

In the following we refer to a CSMA system that employs these parameters as a bound optimized CSMA. The performance of such a system is discussed in the numerical section below.

Proof of Theorem 2. Lower bound A can be written as $\hat{R}_M = \hat{R}_M^M(\lambda^*_c, \delta^*, \beta^*)$, where $\lambda^*_c$ satisfies
\[
\frac{\Phi}{\psi_M} \lambda^*_c \cdot (\delta^*)^{-\kappa} = \eta. 
\] (43)

Using (26) and (41), we have
\[
\eta \Upsilon(\delta^*) = \frac{\lambda^*_c S_{R, \zeta}(\delta^*)}{2}. 
\] (44)
Comparing with (18) and using \( \Upsilon(\delta^*) \leq 1 \) (see (29)) results in \( (\lambda^*_M, \delta^*) \in \tilde{\mathcal{M}}(\eta, \Upsilon(\delta^*)) \). Using the definition, (19), leads to \( \tilde{A}(\eta \cdot \Upsilon(\delta^*)) \geq \tilde{R}^M(\lambda^*_M, \delta^*) \) and the theorem follows. \hfill \Box

To verify that Lower bound B is tighter than Lower bound A, we next prove that the maximal rate, \( \tilde{R}^M_\eta \), is a non-decreasing function of the BP. Since \( \Upsilon(\delta^*) \leq 1 \) we observe that Lower bound B achieves the same bound for a lower value of \( \eta \). Combining with Lemma 2 we conclude that for any BP, \( \eta \), Lower bound B is higher than or equal to Lower bound A.

**Lemma 2:** Given two networks, utilizing the same directional antennas and operating over the same channel, if \( \eta_1 \leq \eta_2 \) then \( \tilde{R}^M_{\eta_1} \leq \tilde{R}^M_{\eta_2} \)

**Proof of Lemma 2.** For the two networks, Lower bound A differs only by the \( \max Q(\zeta, \eta, \alpha) \) expression. Using (23) and comparing (35) with (32) we observe that \( \xi(t) = \mathcal{L}^{-1}\{\Xi(s)\} \) is the probability density function (PDF) of the aggregate interference created by transmitters that their point process is a PPP with density of \( 1/\pi \), and received at a probe receiver guarded by a guard-zone of \( \sqrt{\eta} \). Also, from the integral in (23) we see that an outage event occurs if this aggregate interference is above \( \zeta^{-1/2} \). Clearly, for a fix \( \zeta \), increasing the guard-zone decreases the outage probability. Therefore, denoting \( \zeta_1 = \arg\max_{\zeta} Q(\zeta, \eta_1, \alpha) \) leads to \( Q(\zeta_1, \eta_1, \alpha) \leq Q(\zeta, \eta_2, \alpha) \leq \max_{\zeta} Q(\zeta, \eta_2, \alpha) \), which completes the proof. \hfill \Box

From Lemma 2 we conclude that Lower bound B is tighter than Lower bound A.

IV. INSIGHTS ON THE SPECTRAL EFFICIENCY

A. Decomposition Property of the Lower Bounds

The bounds which were presented in section III can be used to understand the effect of the system parameters and the channel properties on the performance of WANET CSMA. The main advantage of the Lower bounds is that they are composed of several independent parts, each of which is affected by different variables.

The directional antennas affect the ASE bound by a scaling factor, \( \Psi_M \). This factor depends solely on the specific antenna pattern and the channel path loss exponent. The channel fading affects the bound mostly through the scaling factor \( \Phi \), and has a smaller effect through \( U(\beta, \alpha, \rho) \). Conventionally, the rate factor, \( \beta \), can be solved through a 1-dimensional optimization. Furthermore, this optimization only depends on the channel path loss exponent, the fading and distance distributions and the SNR and not on the system parameters (antenna pattern and the maximal BP).

The BP, which is upper bounded by \( \eta \), only affect the ASE through \( Q(\zeta, \eta, \alpha) \) where the parameter \( \zeta \) is also optimized through a 1-dimensional optimization. As was shown in Lemma 2, the ASE increases monotonically with the allowed BP.\(^3\)

\[^3\]The reader should keep in mind that a higher BP can incur a higher system overhead. Due to our prior assumption on the length of the data sub-slot, the calculation of the ASE above does not consider the effect of such additional overhead, and we only limit the maximal back-off probability allowed in the system.

B. Effect of Directional Antennas

To illustrate the effect of directional antennas we consider the following sector antenna model, [22, 29]:

\[
G(\theta) = \begin{cases} 
1 & |\theta| \leq \frac{\pi}{M} \\
1 - \frac{1}{M} & |\theta| > \frac{\pi}{M}
\end{cases}
\]  

where \( M \) is the antenna directivity, which determines the part of space covered by the antenna main lobe (i.e., the main lobe width is \( 2\pi/M \)). This model results in \( \Psi_M = (E[G^\alpha])^{-1} = (E[G(\theta)^\alpha])^{-2} = M^2((M-1)M^{-\kappa} + 1)^{-2} \) (see also [22, 29]). This scaling factor is identical to the scaling factor in the slotted ALOHA case, [29]. Note that for an omnidirectional antenna \( \Psi_1 = 1 \), and for a very narrow main-lobe-width \( \Psi_M \approx M^{2\kappa} \).

C. High SNR Regime

High transmission power results in strong received signals, but also strong interference. This case, which is commonly termed the high SNR regime, is characterized by a negligible contribution of the thermal noise. In order to evaluate the network performance in this case, we consider the limit as \( \rho \to \infty \). In this regime we can simplify the optimization with respect to \( \beta \) for the cases of no fading and a Rayleigh fading channel.

For the case of constant pair-distance (e.g., \( d = 1 \)) and no fading (i.e., \( V = 1 \)) the optimization is solved in closed form:

\[
\beta^* = \arg \max_\beta U(\beta, \alpha, \infty) = \exp \left( \frac{1}{\kappa} + W \left( -\frac{1}{\kappa} e^{-\frac{1}{\kappa}} \right) \right) - 1
\]

where \( W(\cdot) \) is the product-log function also known as the Lambert \( W \) function, which is the inverse of the relation \( f(w) = we^w \).

For the case of constant pair-distance and Rayleigh channel fading (i.e., \( V \sim \text{Exp}(1) \)) the optimization of \( \beta \) can be written
as
\[
\beta^* = \arg \max_{\beta} \beta^{-\kappa} e^{1/\beta} E_i \left(1/\beta\right)
\]
(47)
where \(E_i(z)\) is the exponential integral function, which is the solution to the integral \(\int_0^\infty \frac{e^{-t/z}}{t} dt\). Note that for Rayleigh fading channel \(\Phi = \Gamma \left(1 + \frac{2}{\alpha}\right)\), where \(\Gamma(\cdot)\) is the Gamma function.

D. Low SNR Regime
In the low SNR regime, i.e., \(\rho E[V]E[d^{-\alpha}] \ll 1\), we use the linear approximation of the log function, which results in:
\[
\beta^* = \frac{1}{\alpha} \log(2) \left[ \frac{\rho \beta}{\rho + \beta} \cdot V d^{-\alpha} \right] = \rho \cdot \left(1 - \frac{1}{\kappa}\right)
\]
(48)
and we obtain:
\[
\max_{\beta} U(\beta, \alpha, \rho) = E[V]E[d^{-\alpha}] \cdot \frac{2}{\alpha} \left(\frac{1}{\kappa} - 1\right)^{1-\kappa} \rho^{\alpha-\kappa}.
\]
(49)
Substituting (48) into (40) results in the following optimal interference threshold:
\[
\delta^* = \left(\frac{2}{\alpha - 2}\right) \left(\frac{1}{\eta}\right) \rho^{-1/2}.
\]
(50)
Substituting (50) into (26) leads to the following optimal density of contending pairs:
\[
\lambda_*^c = \left(\frac{2}{\alpha - 2}\right) \left(\frac{\Psi M}{\pi \Phi}\right) \left(\frac{1}{\eta}\right) \rho^{-1/2}.
\]
(51)

E. Small Back-off Probability Analysis
Equation (35) can also be written as:
\[
\Xi(s) = \exp \left(-\frac{1}{\pi} \int_0^\infty (1 - e^{-\alpha r}) 2\pi r dr\right) 
\cdot \exp \left(-\frac{1}{\pi} \int_0^{\sqrt{\pi}} (1 - e^{-\alpha r}) 2\pi r dr\right) 
\cdot \exp \left(-\frac{1}{\pi} \int_0^{\sqrt{\pi}} e^{-\alpha r} 2\pi r dr\right).
\]
(52)
For small enough BP (\(\eta \ll 1\)), we neglect the right hand side integral in (52), which results in:
\[
\Xi(s) \simeq e^\eta \exp \left(-\frac{1}{\pi} \int_0^\infty (1 - e^{-\alpha r}) 2\pi r dr\right).
\]
(53)
Recognizing that the right part of (53) is the characteristic function of the normalized aggregate interference when using the slotted ALOHA protocol, leads us to the following simple relation:
\[
\tilde{A}(\eta) \simeq e^\eta A(0)
\]
(54)
where \(A(0)\) represents the exact ASE of slotted ALOHA WANET. Interestingly, as shown in the numerical section, this approximation is quite good for a large range of BP values.

V. NUMERICAL RESULTS
In the following section we demonstrate the results presented above and compare them to numerical simulations. In this section we assume that all nodes use the sector antenna model, described by (45), and the pairs distance is fixed to \(d = 1\). Fig. 2 presents the maximum sum-rate as function of the allowed BP. The figure presents results for Rayleigh and Rician fading channels with path loss exponents of \(\alpha = 2.7, 3.3, 4\), with omnidirectional antennas (\(M = 1\)) and directional antennas (\(M = 2, 4\)). For the Rician fading case...
we set the Rician K-factor to 1. The figure depicts the ASE evaluated from a slotted CSMA simulation, the performance of a bound optimized CSMA system (described in sub-section III-B), the PPP approximation (using Equation (19)), our two lower bounds and the small BP approximation (using Equation (54)). Clearly, the PPP approximation is very good (less than 20% error for BP up to 0.8). Lower bound B is very tight and therefore describes the slotted CSMA simulation with good accuracy. Lower bound A is a little less tight. However it still predicts the system behavior well, while being quite simple to evaluate. Moreover, the bounds are shown to be very useful for system optimization, as the bound optimized CMAS (applying parameters that were evaluated from Lower bound B) achieves a maximum sum-rate which is very close to the optimal ASE. Since the point of zero back-off probability represents the performance of the slotted ALOHA protocol, Fig. 2 also allows the reader to evaluate the gain of CSMA over ALOHA WANETs. In particular, one can see that the low BP approximation is very useful for all ranges of BP in the figure. Thus, we conclude that the gain of CSMA over ALOHA is very well approximated by $e^\rho$.

Note that the bounds presented in Fig. 2 used the back-off probability approximation (17). Fig. 3 demonstrates the accuracy of this BP approximation. The figure depicts the actual BP when the bound optimized CSMA parameters are applied to a slotted CSMA simulation as a function of the planned maximal BP. As can be seen, the approximation is good, almost up to $P_B = 1$, and all deviations from the planned BP are negligible.

The impact of the SNR, $\rho$, on Lower bound B is shown in Fig. 4. The impact of $\rho$ is independent of the antenna pattern, and hence the figure is depicted for omnidirectional antenna ($M = 1$). As can be seen, Lower bound B reaches a saturation in the high SNR regime while in the low SNR regime it is proportional to $\rho^{1-\alpha}$ (see (49)). In the noise limited regime the impact of the fading is given by the inverse of $\Phi$, which equals to $\Phi^{-1} = 1.09, 1.12, 1.13$ for $\alpha = 2.7, 3.3, 4$ respectively.

The optimal system parameters, $\zeta^*$ and $\beta^*$ are presented in Fig. 5 and Fig. 6 respectively. As is apparent, the use of (40) enables the generation of these curves quite easily. These curves are very useful for CSMA network optimization, and can lead to more efficient network deployment. In Fig. 5, the dashed curve represents the line $\zeta^* = P_B$ which is equivalent to $\delta = \beta^{-1}$ (see (38)). The intersection of the $\zeta^* = P_B$ line with the $\zeta^*$ curves divides the $\zeta^*$ curves into two regions. In the left region a single interferer can cause an outage event, while in the right region, which seems to be more practical for CSMA WANETs, only the aggregate interference from two or more interferers can result in an outage event.

VI. CONCLUSIONS

In this paper we analyzed the performance of two-phase CSMA WANETs with general directional antenna model, general pair-distance distribution and a general channel fading model. We constructed a good approximation model, which use a back-off approximation and an interference approximation, and employed the ASE of this model as a performance measure. We then presented useful simplified lower bounds that give a good prediction of the ASE of CSMA WANETs. These bounds are very convenient for the evaluation and optimization of CSMA WANETs performance. The first lower bound provides insights on the effect of antenna patterns and channel properties on the ASE. For example it shows that for a sector antenna model, directional antennas with narrow main-lobe increase the ASE by a factor of $M^{-\frac{1}{2}}$. The second lower bound was shown to produce an even better approximation of the ASE. This bound is also very effective for predicting the optimal system parameters. We verified through simulations that the use of these parameters (including the users' rates, interference threshold and users density) results in performance that is very close to the maximal system performance.

In the low SNR regime we showed that the ASE of WANETs is proportional to $\rho^{1-\alpha}$, where $\rho$ is the SNR. This result differs from the point-to-point case, in which the capacity is proportional to $\rho$. The performance improvement in
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Chapter 3

The Ergodic Rate Density of Poisson Wireless Ad-Hoc Networks

The ergodic rate density of Poisson wireless ad hoc networks

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Abstract—In recent years, much attention has been paid to the analysis of random wireless ad hoc networks (WANETs) that combine the effect of the physical layer and the medium access layer. However, most works have concentrated on an outage rate model which does not accurately describe the performance of modern communication systems. In this work we consider the ergodic rate density (ERD) of a random ALOHA WANET with a homogeneous Poisson point process node distribution. We present two novel lower bounds on the ERD, one for general transmission and reception strategies and the other for receivers with a spatial interference cancellation capability. The bounds are simpler than previously published results for ALOHA WANETs (that have primarily considered the outage rate density). In addition, the bounds and the bounding technique are quite general and enable the derivation of closed form expressions of the ERD for various network models. The efficiency and simplicity of the bounds are demonstrated through several applications, and insights are drawn on the behavior of the network performance as function of the path-loss factor, transmission strategies and number of antennas. Simulation results demonstrate that these simple lower bounds predict the performance of ALOHA WANETs with high accuracy.

I. INTRODUCTION

Wireless ad hoc networks (WANETs) offer simplicity and flexibility, which make them suitable for many practical applications. These networks do not depend on infrastructure such as base stations, and are typically coordinated by decentralized multiple access protocols. The ALOHA protocol, [2], is the simplest packet-based access mechanism. Together with its modified versions (e.g., [3], [4]) the ALOHA access protocol has attracted much attention both in theory and in practice.

While some interesting works on the capacity of WANETs have considered specific network structures (e.g., [5]), more general insights have been obtained from the analysis of random networks. The most popular model for the positions of active users in random WANETs is the homogeneous Poisson Point Process (PPP), [3]. In this model, the users’ locations are assumed to be uniformly distributed over an infinite plane. The PPP model enables the analysis of WANET performance, and formulates the performance dependence on user density, without having to take the specific users’ locations into account. In this work we use a PPP model to analyze the WANET performance. We focus on the access protocol and the physical layer processing, and use simplifying assumptions on the operation of higher communication layers.

Most previous analyses of WANETs have focused on the concept of outage capacity. In this approach, the transmitters encode each message using an error correction code, and transmit it (once or several times). The analysis is based on the calculation of the probability that the receiver will be able to decode the message correctly. If the receiver cannot decode the message, it is said to be in ‘outage’. For example in the context of PPP, the popular Transmission Capacity (TC) measure, [6], is defined as the maximal area spectral efficiency subject to a fixed outage probability.

The effect of transmission strategies on the TC of ALOHA WANETS under fading channels was studied in [7] for fixed transmission power, channel inversion and opportunistic ALOHA. Using upper and lower bounds on the outage probabilities they showed that the channel inversion scheme is inferior to the fixed transmission power scheme.

Upper and lower bounds on the TC of an ad hoc network when each node is equipped with multiple antennas were derived in [8]. Although they considered the transmission of multiple streams by each transmitter, the optimal number of transmitting streams was shown to be one. Assuming channel state information in the transmitter, the best scaling of the transmission capacity with the number of antennas was achieved by transmit beamforming combined with interference cancellation in the receiver. In this case, the TC was shown to scale linearly with the number of antennas per node.

However, generally speaking, the analysis of the network TC involves quite complicated expressions. In order to make the TC analysis tractable, simplifying assumptions are commonly used; e.g., small outage [9], specific fading models [3] or specific path-loss factors [10].

An alternative approach, adopted here, evaluates the ergodic rate density (ERD) using the maximum mutual information between the transmitted signal and the received signal given the interferers activity, [1]. Basic results in information theory guarantee that such a rate, which is higher than the outage rate, is indeed achievable although it may incur significant delays (e.g., [11]).

An example of the use of the ergodic rate as a performance metric can be found in [5] where the authors developed a mathematical framework for the ergodic capacity region of specific topology WANETs under several transmission protocols. Haenggi, [12], introduced a lower bound on the ERD of users in ALOHA WANET. This bound was developed for the special case in which both the desired and interference channels fading follow the Rayleigh distribution. Stamiotti et al. [13], introduced upper and lower bounds on the performance of WANETs applying frequency hopping scheme. Their framework included Rayleigh fading channels and the ERD was evaluated under the assumption of no channel
state information (CSI) on the interfering transmitters\(^1\). The resulting lower bound was shown to be tight for low user densities.

An asymptotic upper bound on the ERD of WANETs equipped with \(N\) receive antennas was introduced in [14]. In their model they assumed that the interferers keep a minimum distance from receivers. The SINR was shown to grow as \(N^{\alpha/2}\) where \(\alpha\) is the path-loss factor and \(N\) is the number of antennas.

Note that there is a direct relationship between the expressions of the Outage-Rate Density (ORD) and the ERD. To be more precise, the outage probability describes the Cumulative Density Function (CDF) of the Signal to Interference plus Noise Ratio (SINR). Thus, the ERD can be evaluated as an integral of the achievable rates with respect to the CDF of the SINR. However, in most cases, closed form expressions of the outage probability are not available, and the integration to obtain the ERD is even more difficult. Thus, it is quite difficult to obtain insights on the ERD from the known ORD expression. The work presented below illustrates that in the general case the characterization of the ERD may be even simpler than the analysis of the ORD.

In this paper we analyze a random PPP WANET utilizing the ALOHA protocol, and present novel lower bounds on its ERD. The first bound holds for any reception strategy, while the second bound holds for receivers that apply spatial interference cancellation. Both bounds are much simpler than equivalent results obtained for the outage rate metric (e.g., TC). Nevertheless, these bounds are very close to the actual ERD, and can even predict the behavior of ORD as function of the network parameters; e.g., path-loss factor, network density and number of antennas per node. The usefulness of the bounds is illustrated by several common applications: For the single antenna case, the applications include fixed transmission power, channel inversion and opportunistic ALOHA. For multiple antenna WANETs, we present cases of transmit beamforming with or without interference cancellation in the receiver. The tightness of the derived bounds is also evaluated and shown by simulation.

The ERD analysis have three distinct advantages, which will be emphasized in what follows. First, the ergodic rate is achievable and higher than the outage rate. Second, this analysis results in a much simpler performance bounds, and third, its behavior is very similar to the outage rate, so that the derived results can also predict the general behavior in outage models.

The rest of this paper is organized as follows: The following section presents various schemes that can achieve the ERD. Section III describes the system model. Section IV introduces the novel lower bounds on the ERD. Section V demonstrates the application of these bounds to five different scenarios and introduce further insights on the bounds. Section VI presents our concluding remarks.

---

\(^1\)This CSI assumption describes well system that use fast frequency hopping. In this paper we consider the complete CSI case, which can characterize slow/medium frequency hopping systems.
mobility is significant. On the other hand, the HARQ schemes achieve the network ERD by allowing each user to transmit data at a different rate (that matches its instantaneous conditions). Thus, the ERD can be achieved in the network with very short delays, and longer delays are required only to achieve fairness between users.

III. SYSTEM MODEL

We assume a decentralized wireless ad hoc network utilizing an ALOHA protocol (e.g., [3]). For the position of nodes we use the popular Poisson bipolar network model [24], [25]. The network in this model consists of a PPP of active transmitters and a set of receivers that are positioned at fixed distance, \( d \), and at a uniformly randomly chosen orientation from their paired transmitters. The density of the two dimensional PPP of the active transmitters is given by \( \lambda \).

The desired power, received at receiver \( i \) from its paired \((i\text{-th})\) transmitter is given by:

\[
S_i = \rho(Y_i)d^{-\alpha}Y_i
\]

where \( Y_i \) is the effective power fading between the \( i \)-th transmitter and its desired \( i \)-th receiver. This effective power fading represents any random change in the power of the received signal, for example due to the transmitter preprocessing, channel fading and receiver postprocessing.

We assume that the transmission power can be adapted according to the power gain of the desired channel, following some predetermined transmission policy. For ease of notation, we mark the transmitted power by \( \rho(Y_i) \), where the function \( \rho \) describes the transmission strategy. The path-loss factor is denoted by \( \alpha > 2 \).

The power received at receiver \( i \) from transmitter \( j \) for \( i \neq j \) is:

\[
W_{i,j} = \rho(Y_j)X_{i,j}^{-\alpha}V_{i,j}
\]

where \( V_{i,j} \) is the effective power fading and \( X_{i,j} \) is the distance between the \( i \)-th transmitter and the \( j \)-th receiver respectively. The fading variables \( V_{i,j} \) are independent and identically distributed (i.i.d) and statistically independent of all distance variables. In the following we also use the notation \( V \) and \( Y \) when we discuss the statistical nature of one of the Random Variables (RVs) \( V_{i,j} \) and \( Y_{i,j} \), bearing in mind that these are single representatives of families of iid RVs. Note that in some cases the distribution of \( Y \) may be identical to the distribution of \( V \), whereas in other cases these two RVs may have different distributions (as will be shown in section V).

We use the shift invariant property of the system, [26], to analyze the performance of the network using user 0 as a probe receiver. Without loss of generality, we assume that the probe receiver is located at the origin.

For notational simplicity, in the following we drop the probe receiver index and define the set \( \mathcal{A} \) as the set of all active pairs, excluding the probe pair. The aggregate interference, measured at the probe receiver, can be written as:

\[
I = \sum_{j \in \mathcal{A}} W_j = \sum_{j \in \mathcal{A}} \rho(Y_j)X_{j}^{-\alpha}V_j
\] and the power of the desired signal, measured at the probe receiver, can be written as: \( S = \rho(Y) \cdot d^{-\alpha} \).

Without loss of generality we assume that \( d = 1 \). Denoting by \( \sigma^2 \) the contribution of the thermal-noise, the signal-to-interference-and-noise-ratio (SINR) measured at the probe receiver is given by \( \frac{S}{\sigma^2 + 1} \). The ergodic rate density (ERD) of a network with an active user density of \( \lambda \) is given by:

\[
R(\lambda) = \lambda \cdot E \left[ \log_2 \left( 1 + \frac{S}{\sigma^2 + 1} \right) \right]
\]

IV. LOWER BOUNDS ON THE ERD

In this section we present two lower bounds on the ERD of ALOHA WANETS. Theorem 1 presents a bound that holds in general, whereas Theorem 2 presents a tighter bound for the specific case of WANETs with spatial interference cancellation capabilities.

Theorem 1 (General case): A lower bound on the ERD of a network with an active user density of \( \lambda \) is:

\[
R(\lambda) \geq R_{LB}(\lambda)
\]

where

\[
R_{LB}(\lambda) = \lambda e^{2} \cdot E \left[ \log_2 \left( 1 + \frac{\rho(Y) \cdot Y}{\sigma^2 + C_{\alpha,M} \cdot \lambda^2} \right) \right]
\]

and

\[
C_{\alpha,M} \triangleq \frac{2}{\alpha(\alpha - 2)^2} \left( \frac{\pi \alpha E \left[ V^2 \right]}{E \left[ \rho^2 \left( Y \right) \right]} \right)^{\frac{\alpha}{2}}
\]

The proof of the lower bound is given in appendix A below and is based on Jensen’s inequality. However, since the expectation on the aggregate interference in ALOHA WANETs is infinite, the derivation is split into two cases. The cases are defined by the power received from the strongest interferer, which is above or below a predefined threshold parameter. In the first case, the rate is lower bounded by 0, and in the second, the rate is lower bounded using Jensen’s inequality. The power threshold parameter is then optimized to achieve a tighter bound.

Theorem 2 (Spatial interference cancellation): A lower bound on the ERD of a network with an active user density of \( \lambda \), when each receiver cancels its closest transmitters and \( M \geq \left\lceil \frac{\alpha}{2} \right\rceil \) is:

\[
R^{M}(\lambda) \geq R_{LB}^{M}(\lambda)
\]

where

\[
R_{LB}^{M}(\lambda) = \lambda \cdot E \left[ \log_2 \left( 1 + \frac{\rho(Y) \cdot Y}{\sigma^2 + C_{\alpha,M} \cdot \lambda^2} \right) \right]
\]

and for \( 2 < \alpha < 4 \)

\[
C_{\alpha,M} \triangleq \frac{2\pi^2}{\alpha - 2} E \left[ V \right] E \left[ \rho(Y) \right] (M - \frac{\alpha}{4})^{1 - \frac{\alpha}{2}}
\]

while for \( 4 \leq \alpha < 6 \)

\[
C_{\alpha,M} \triangleq \frac{2\pi^2}{\alpha - 2} E \left[ V \right] E \left[ \rho(Y) \right] \left( M - \frac{1}{2} - \frac{\alpha}{4} \right)^{2 - \frac{\alpha}{2}} \frac{1}{M - 1}
\]

The proof of Theorem 2 is given in appendix B below. The proof is again based on Jensen’s inequality, but this time
using the fact that in this case the expectation of the aggregate interference is bounded. The next corollary shows that theorem 2 is tight for large number of cancelled transmitters (for small number of cancelled transmitters, a tighter bound can be obtained by combining the bounding techniques of Theorem 1 and Theorem 2).

**Corollary 1:** The lower bound on the ERD from Theorem 2 converges to the network ERD when the number of cancelled transmitters grows to infinity:

$$\lim_{M \to \infty} \frac{R_{LB}^C(\lambda)}{R^M(\lambda)} = 1.$$  \hfill (12)

The proof of Corollary 1 is given in appendix C below.

**V. APPLICATIONS OF THE LOWER BOUNDS**

In this section we apply the theorems of Section IV to various scenarios. The obtained bounds for the different applications are also discussed, evaluated and compared to simulation results. At the end of the section we present several insights on the impact of the system parameters and the network policies on the ERD.

**A. Single Antenna WANETs**

In the single antenna case we have no pre/post-processing. Thus, the distribution of $Y$ is identical to the distribution of $V$, and we assume that both follow an exponential distribution ($\sqrt{V}$ and $\sqrt{Y}$ have a Rayleigh distribution). Due to power normalization we set the fading power to $E[V] = E[Y] = 1$ (i.e., $V, Y \sim \text{Exp}(1)$). In this case, the expectation over $V$ in (7) is:

$$E\left[V^{\frac{\alpha}{2}}\right] = \Gamma\left(1 + \frac{2}{\alpha}\right)$$ \hfill (13)

where $\Gamma(.)$ represents the Gamma function.

1) **Single Antenna - Fixed Transmission Power:** Consider a single antenna WANET that applies a fixed transmission power strategy. The fixed transmission power is applied by $\rho(Y) = \rho_0$. Substituting (13) into Theorem 1 results in:

$$R_{LB}^C(\lambda) = \lambda e^{\frac{\alpha}{2} - 1} \cdot E\left[\log_2\left(1 + \frac{1}{\rho_0} \frac{Y^{\alpha}}{\sigma^2 + C_\alpha^C \cdot \lambda^{\frac{2}{\alpha}}}\right)\right]$$

$$= \frac{\lambda}{\ln(2)} \cdot \exp\left(\frac{2}{\alpha} - 1 + \frac{1}{\rho_0} \left(\sigma^2 + C_\alpha^C \cdot \lambda^{\frac{2}{\alpha}}\right)\right) \cdot E_i\left(\frac{1}{\rho_0} \left(\sigma^2 + C_\alpha^C \cdot \lambda^{\frac{2}{\alpha}}\right)\right)$$ \hfill (14)

where

$$C_\alpha^C \equiv \frac{2}{\alpha (\alpha - 2) \Gamma}\left(\pi \alpha \Gamma\left(1 + \frac{2}{\alpha}\right)\rho_0^{\frac{2}{\alpha}}\right)^{\frac{\alpha}{2} - 1}$$ \hfill (15)

and $E_i(z)$ is the exponential integral function, which is the solution to the integral $\int_z^{\infty} \frac{e^{-t}}{t} \, dt$.

The ERD and the bound of (14) are depicted in Fig. 1 as a function of the active user density, $\lambda$. As can be seen, in this case the bound is about 30% below the ERD. Note also that the bound exhibits the same behavior as the ERD; hence it is useful to draw insights from its behavior. As a reference, Fig. 1 also depicts the lower bound introduced by Stamatiou et al., [13, eq. 26]. This bound was derived for the case of partial CSI. Yet, it is also a lower bound in the complete CSI case described herein. The Stamatiou bound is even tighter then the bound presented herein for very low user densities. Yet, for medium to high user densities, and in particular near the optimal user density, the bound derived herein is more useful.

**Note on simulations:**

The results are plotted using Monte-Carlo simulations and were averaged over 50,000 Network realizations. All simulations were performed in the interference limited regime in which the contribution of the thermal noise can be neglected. The channel was characterized by Rayleigh fading and for most of the simulations the path-loss factor was $\alpha = 3$.

In most figures we also evaluated (as a reference) the performance of an outage scheme, in which each packet contained a code word of fixed length. In this case the performance metric was the ORD, given by:

$$R_{out}(\lambda) = \max_\beta \lambda \cdot P_{out}(\lambda, \beta) \cdot \log_2(1 + \beta)$$ \hfill (16)

where, [27, eq. 5.83],

$$P_{out}(\lambda, \beta) \triangleq \Pr\left(\sum_{k=1}^{N_s} \log_2\left(1 + \frac{S}{\sigma^2 + I_k}\right) \geq N_s \cdot \log_2(1 + \beta)\right),$$ \hfill (17)

$N_s$ is the number of symbols in a packet, $S$ and $\sigma^2$ are the signal and noise powers, and $I_k$ is the interference power, experienced at the $k$-th symbol. Several works, [28]–[30], showed that (17) describes the behavior of modern error-correction codes.

Fig. 2 depicts the maximal rate densities (i.e., using the optimal user density) as function of the path-loss factor, $\alpha$, for the fixed transmission power strategy. The figure depicts the ERD, the lower bound introduced by Haenggi [12, Eq. 45], the

\footnote{Unlike the ERD, the ORD is different for slotted and unslotted ALOHA. In slotted ALOHA the interference is constant within each slot, and the sum in (17) can refer to slots instead of symbols. In the following simulations we assume slotted ALOHA with one slot per packet; i.e., we used $N_s = 1$.}
of the desired channel is $f_Y(y)$ and the path-loss factor. The lower bound introduced by Haenggi [12] is tighter for large $\alpha$ and looser for smaller $\alpha$. Note that the bound in [12] is valid only for Rayleigh fading channels and WANETs applying fixed transmission power.

2) Single antenna - Opportunistic ALOHA Strategy: Consider a WANET with single antenna nodes, that apply opportunistic ALOHA strategy with a channel threshold of $Y_t$ [31]. The transmission strategy is described by:

$$
\rho(Y) = \begin{cases} 
\rho_0 & \text{if } Y \geq Y_t \\
0 & \text{Otherwise}
\end{cases}
$$

(18)

For the Rayleigh fading case, the probability of transmission is $\Pr(Y > Y_t) = e^{-Y_t}$ and the probability density function of the desired channel is $f_Y(y|Y > Y_t) = e^{-y-Y_t}$. Using Theorem 1, the lower bound can be written as:

$$
R_{LB}^T(\lambda) = \lambda e^{-\sigma^2-Y_t} \cdot E \left[ \log_2 \left( 1 + \frac{Y}{K_{\alpha,Y_t}} \right) | Y > Y_t \right]
$$

$$
= \lambda e^{-\sigma^2-Y_t} \log_2 \left( 1 + \frac{Y_t}{K_{\alpha,Y_t}} \right)
$$

$$
+ \frac{e^{K_{\alpha,Y_t}+Y_t}}{\ln(2)} \cdot E_i(K_{\alpha,Y_t} + Y_t)
$$

(19)

where

$$
K_{\alpha,Y_t} \triangleq \frac{1}{\rho_0} \left( \sigma^2 + C_{\alpha,Y_t}^T \cdot \lambda^\frac{\alpha}{2} \right)
$$

(20)

and

$$
C_{\alpha,Y_t}^T \triangleq \frac{2}{\alpha (\alpha - 2)^\frac{\alpha}{2}} \left( \pi \alpha \Gamma \left( 1 + \frac{2}{\alpha} \right) \rho_0 e^{-Y_t} \right)^\frac{\alpha}{2}.
$$

(21)

For the last curve we used the upper bound expression on the outage probability as function of the SIR threshold ($\beta$) given in [7]. Then we optimized the SIR threshold parameter as described in (16).

Using $E_i(z) > \frac{1}{2} e^{-z} \ln \left( 1 + \frac{2}{z} \right)$, [32], the lower bound in (19) can be simplified to:

$$
R_{LB}^T(\lambda) = \frac{1}{2} \lambda e^{-\sigma^2-Y_t} \log_2 \left( K_{\alpha,Y_t}^T + Y_t \right)
$$

$$
+ \log_2 (K_{\alpha,Y_t}^T + Y_t + 2) - 2 \log_2 \left( K_{\alpha,Y_t}^T \right)
$$

(22)

which is very close to (19) for large values of $Y_t$.

Fig. 3 depicts three rate densities as function of the channel threshold parameter, $Y_t$, for an active user density of $\lambda = 0.26$. The rate densities are the ERD, the lower bound described in (22) and the ORD. In this case, the difference between the ERD and its bound is around 25%, and again both curves show the same behavior. In particular, both the ERD and its lower bound have a maxima around $Y_t = 1$, which implies that (22) can also be used to find the optimum threshold for WANET when applying opportunistic ALOHA mechanism. The ORD is somewhat lower from the lower bound (on the ERD). The ORD of the opportunistic ALOHA scheme was discussed in [7], but without any closed form expression.

3) Single antenna - Channel Inversion Strategy: We next analyze the case of a single antenna with a channel inversion power control strategy [7]. Each transmitter adapts the transmission power to achieve a fixed received power in its destination receiver; i.e. $\rho(Y) = \rho_0 Y^{-1}$ where $\rho_0$ is a constant. This strategy leads to:

$$
E \left[ \rho Y \left( Y \right) \right] = \rho_0^2 \cdot E \left[ Y^{-\frac{\alpha}{2}} \right].
$$

(23)

Substituting (23) into Theorem 1 leads to the following lower bound:

$$
R_{LB}^T(\lambda) = \lambda e^{-\sigma^2-Y_t} \cdot \log_2 \left( 1 + \frac{1}{\rho_0} \left( \sigma^2 + C_{\alpha,Y_t}^T \cdot \lambda^\frac{\alpha}{2} \right) \right)
$$

(24)

where

$$
C_{\alpha,Y_t}^T \triangleq \frac{2}{\alpha} \left( \pi \alpha \Gamma \left( 1 + \frac{2}{\alpha} \right) \rho_0^2 e^{-Y_t} \right)^\frac{\alpha}{2}.
$$

(25)

and for Rayleigh fading

$$
E \left[ Y^{-\frac{\alpha}{2}} \right] = \Gamma \left( 1 + \frac{2}{\alpha} \right),
$$

$$
E \left[ Y^{-\frac{\alpha}{2}} \right] = \Gamma \left( 1 - \frac{2}{\alpha} \right).
$$

(26)
In the interference limited regime, in which the thermal noise can be neglected, we can also write a closed form expression for the optimal user density:

$$\lambda^* = \arg \max_\lambda R_{LB}(\lambda)$$

$$= \left(\frac{\alpha - \beta}{\alpha \sigma^2}\right) \frac{Y^\frac{1}{\beta}}{E[Y]} \frac{W(\frac{\alpha}{2} - \frac{\beta}{2} e^{-\frac{\alpha}{2}}) - 1}{E[Y] - 10^{\frac{\alpha}{2}}}(27)$$

where $W(\cdot)$ is the product-log function also known as the Lambert $W$ function, which is defined as the inverse function of $f(w) = we^w$. The lower bound in this case can be written as:

$$R_{LB}^B(\lambda^*) = \frac{e^{\frac{\alpha}{2} - 1} \left(\frac{\alpha}{2} + W(-\frac{\alpha}{2} e^{-\frac{\alpha}{2}})\right)}{\ln(2) \cdot E[Y]} - \frac{(\alpha - \beta)}{(\beta / 2)} \frac{\Gamma(N_T - \frac{\beta}{2})}{\Gamma(N_T)} (28)$$

Note that the equivalent closed form expression for the ORD metric is unknown, and in [7] only bounds on its outage probability were derived. Fig. 1 depicts the ERD, the lower bound, (24), and the ORD for this case. An analytical comparison between the performance of the fixed transmission power and the channel inversion strategies is presented in V-C2.

**B. Multi antenna WANETs**

1) **Transmit Beamforming:** In this subsubsection we analyze the case of transmit beamforming with $N_T$ antennas and a single receive antenna. We also assume the use of the simple channel inversion power control strategy (i.e., $\rho(Y) = \rho_0 Y^{-\frac{1}{2}}$ where $\rho_0$ is a constant). The preprocessing of beamforming over $N_T$ antennas results in a Chi-square desired channel distribution, with $2N_T$ degrees of freedom, $2Y \sim \chi^2_{2N_T}$, which leads to:

$$E[Y] = \rho_0^\frac{1}{2} \cdot E[Y - \frac{1}{2}] = \rho_0^\frac{1}{2} \Gamma(N_T - \frac{\beta}{2}) \Gamma(N_T).$$

This preprocessing does not change the interference channel statistics, $V$. Substituting (29) into Theorem 1 leads to:

$$R_{LB}^B(\lambda) = \lambda e^{\frac{\alpha}{2} - 1} \log_2 \left(1 + \frac{1}{\rho_0^2} (\sigma^2 + C^B_\alpha \cdot \lambda^2) \right) (30)$$

where

$$C^B_\alpha = \frac{\alpha}{\beta} \left(\frac{\pi^2 \Gamma(1 + \frac{\beta}{2}) \Gamma(N_T - \frac{\beta}{2})}{\Gamma(N_T)} \right)^2. (31)$$

In the interference limited regime we can also write a closed form expression for the optimal user density:

$$\lambda^* = \left(\frac{\alpha - \beta}{\alpha \sigma^2}\right) \frac{Y^\frac{1}{\beta}}{E[Y]} \frac{W(\frac{\alpha}{2} + W(-\frac{\alpha}{2} e^{-\frac{\alpha}{2}}) - 1)}{\Gamma(1 + \frac{\beta}{2}) \Gamma(N_T - \frac{\beta}{2})} (32)$$

where $W(\cdot)$ is the product-log function as defined in subsection V-A3. The lower bound in this case can be written as:

$$R_{LB}^B(\lambda^*) = \frac{e^{\frac{\alpha}{2} - 1} \left(\frac{\alpha}{2} + W(-\frac{\alpha}{2} e^{-\frac{\alpha}{2}})\right)}{\ln(2) \cdot \Gamma(1 + \frac{\beta}{2}) \Gamma(N_T - \frac{\beta}{2})} - \frac{(\alpha - \beta)}{(\beta / 2)} \frac{\Gamma(N_T - \frac{\beta}{2})}{\Gamma(N_T)} (33)$$

Using Kershaw’s inequality, [33], [34],

$$\Gamma(N_T) \Gamma(N_T - \frac{\beta}{2}) \geq \left(N_T - \frac{1}{2} - \frac{1}{\alpha}\right)^\frac{\beta}{2} (34)$$

and substituting into (33) simplifies the lower bound to:

$$R_{LB}^B(\lambda^*) \geq \frac{e^{\frac{\alpha}{2} - 1} \left(\frac{\alpha}{2} + W(-\frac{\alpha}{2} e^{-\frac{\alpha}{2}})\right)}{\ln(2) \cdot \Gamma(1 + \frac{\beta}{2}) \Gamma(N_T - \frac{\beta}{2})} - \frac{(\alpha - \beta)}{(\beta / 2)} \frac{\Gamma(N_T - \frac{\beta}{2})}{\Gamma(N_T)} (35)$$

Fig. 4 depicts the ERD, the lower bound described in (30) and the ORD as function of the active user density, $\lambda$, for the case of transmit beamforming with channel inversion and 1, 3 and 9 transmit antennas. As the figure illustrates, the lower bounds for these three cases describe the network performance very well as function of both the density and the number of transmit antennas. As expected, the optimal density of users increases with the number of antennas.

Note that Theorem 1 can also be applied to the case of receive diversity or combined receive and transmit diversity, by using the relevant probability function of the desired channel $RV$, $Y$ (in this case, the interference fading distribution does not change). In particular, the case of single transmit antenna and $N_R$ receive antennas, results in the same ERD , (30), substituting only $N_T$ by $N_R$.

2) **Transmit Beamforming and Interference Cancellation:** We assume $N_T \geq 2$ transmit antennas and $N_R \geq [\alpha/2]$ receive antennas for each node. The transmitter performs beamforming and channel inversion as described in subsubsection V-B1. Each receiver uses its antennas to cancel its $N_R - 1$ closest interferers.
Substituting (40) into (37) leads to:

\[ \text{Defining } C_2 \text{ as shown in Corollary 1.} \]

The transmission power can be written as \( \rho(Y) = \rho_0 Y^{-1} \).

For the Rayleigh fading channel, \( 2Y \) again has a chi-square distribution with \( 2N_T \) degrees of freedom and

\[
E[\rho(Y)] = \rho_0 \cdot E[Y^{-1}] = \frac{\rho_0}{N_T - 1}. \tag{36}
\]

Defining \( C_{\alpha,N_R,N_T} \) as shown in Corollary 1.

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In the interference limited regime we can also optimize (37) and obtain a closed form expression for the optimal user density:

\[
\lambda^*_S = \left( K_{\alpha,N_R,N_T}^{-1} \left( \exp \left( \frac{\alpha}{2} + W \left( -\frac{\alpha}{2} e^{-\frac{\alpha}{2}} \right) \right) - 1 \right) \right)^{-\frac{1}{\alpha}}. \tag{38}
\]

Substituting (40) into (37) leads to:

\[
R_{\text{LB}}^S \left( \lambda^*_S \right) = \frac{2\pi^2}{\ln(2)} \left( \frac{\sigma^2 + C_{\alpha,N_R,N_T} \lambda^\frac{1}{\alpha}}{\alpha - \lambda} \right)^{\frac{1}{\alpha}}. \tag{39}
\]

Fig. 5 depicts the rate densities for transmit beamforming and interference cancellation at the receiver, as function of the active user density, \( \lambda \). For this simulation we used \( N = N_T = N_R \). The curves of the ERD, the lower bound, (37), and the ORD, are plotted for the cases of \( N = 3, 6 \) antennas. As a reference we added the sum-rate curve of the single antenna channel inversion, (24). Note that the accuracy of the lower bound, (37), is indeed improving with the number of antennas as shown in Corollary 1.

C. Further Insights from the Lower Bounds

1) Impact of the path-loss factor on the ERD: As shown below, the lower bound on the maximal ERD for the channel inversion scheme in the interference limited regime, (28), can be approximated for large values of \( \alpha \) by:

\[
R_{\text{LB}}^I \left( \lambda_i^* \right) \approx \frac{\sigma^2 + \log \left( \frac{\alpha}{2} \right) + \frac{\lambda}{\alpha} \log \left( \frac{\lambda}{2} \right)}{\pi e^2 \cdot \ln(2) \cdot E \left[ V^\frac{\alpha}{2} \right]} \cdot E \left[ V^{-\frac{\alpha}{2}} \right]. \tag{42}
\]

This approximation shows the dependence of the maximal ERD in the channel model when the density of active users is optimized. In particular, it shows that the effect of the path-loss factor is more complicated, but, for large values of \( \alpha \) the maximal ERD grows linearly with \( \alpha \).

The accuracy of (42) is depicted in Fig. 6, where the two upper curves depict the exact ERD, (28), and its approximation, (42), for the case of a single antenna and Rayleigh fading. As can be observed, this simple approximation captures the effect of \( \alpha \) on the maximal ERD with high accuracy.

The approximated expression, (42), is derived from (28) through the following steps:

\[
R_{\text{LB}}^I \left( \lambda_i^* \right) \overset{(a)}{=} \frac{\sigma^2 + \log \left( \frac{\alpha}{2} \right) + \frac{\lambda}{\alpha} \log \left( \frac{\lambda}{2} \right)}{\pi e^2 \cdot \ln(2) \cdot E \left[ V^\frac{\alpha}{2} \right]} \cdot E \left[ V^{-\frac{\alpha}{2}} \right] \tag{43}
\]

where we assumed large values of \( \alpha \) and (a) used \( \left| W \left( -\alpha/2e^{-\frac{\alpha}{2}} \right) \right| \ll \alpha/2 \) (and hence we omitted the Lambert \( W \) terms) and (b) used \( 1 - 2/\alpha \approx e^{-\frac{\alpha}{2}} \) and \( e^{\frac{\alpha}{2}} - 1 \approx e^{\frac{\alpha}{2}} \). Using the three dominant terms of the Maclaurin series of \( x^{-x} \) for \( x = 2/\alpha \) results in

\[
\left( \frac{\alpha}{2} \right)^{\frac{\alpha}{2}} \approx 1 - \frac{2}{\alpha} \log \left( \frac{\alpha}{2} \right) + \frac{\alpha}{2} \log^2 \left( \frac{\alpha}{2} \right) \tag{44}
\]

which together with (43) led to (42).

Note that this growth of the ERD in \( \alpha \) is guaranteed only for the maximal ERD, i.e., where the density of active users is optimized. This conclusion does not hold for an arbitrary selection of active user density. For some user densities the ERD even decreases with the rise in \( \alpha \) as shown by the following Proposition:

**Proposition 1:** For any \( \alpha > 2 \) there exists \( \lambda_0 \) such that

\[
\left. \frac{\partial R_{\text{LB}}^I(\lambda)}{\partial \lambda} \right|_{\lambda=\lambda_0} < 0.
\]

**Proof of Proposition 1:** The lower bound (6) is the multiplication of two terms. The left expression, \( \lambda e^{\frac{\alpha}{2}-1} \), clearly decreases with \( \alpha \). The right expression, which can be written as

\[
E \left[ \log_2 \left( 1 + \frac{\rho(Y)}{\sigma^2 + x} \right) \right]_{x=C_\alpha \lambda^\frac{1}{\alpha}} \tag{45}
\]

decreases in \( x \). Selecting \( \lambda \) that will guarantee that \( \left. \frac{\partial (C_\alpha \cdot \lambda^\frac{1}{\alpha})}{\partial \alpha} \geq 0 \) will ensure that (45) is decreasing with
This can be obtained by selecting the active user density to be \( \lambda_0 \geq \exp \left( -\frac{\partial C_\alpha}{\partial \alpha} \cdot \frac{2}{\alpha} \right) \) (which is positive and finite for any \( \alpha > 2 \)).

The lower curve in Fig. 6 depicts the lower bound (24) for Rayleigh fading and \( \lambda = 3 \) as function of the path-loss factor. As can be noticed this curve is not monotonic, and actually decreases for \( \alpha > 3 \).

2) Fixed transmission Power versus channel inversion: In the following subsection we compare the lower bound of the fixed transmission power, (14), to the lower bound of the channel inversion strategy, (24), for any active user density.

**Lemma 1:** For a Rayleigh fading channel, using fixed transmission power gives a higher ERD than channel inversion, for any active user density:

\[
R_{LB}^f(\lambda) > R_{LB}^\alpha(\lambda)
\]  

**Proof of Lemma 1:** Define:

\[
g(x) \triangleq \frac{R_{LB}^\alpha((x/C_\alpha)^{2/\alpha}) - R_{LB}^f((x/C_\alpha)^{2/\alpha})}{\ln x} = e^x \cdot E_1(x) - \ln \left( \frac{\Gamma \left( \frac{1}{2} \right)}{\Gamma \left( \frac{1}{2} - \frac{2}{\alpha} \right)} \right)^{-\frac{1}{2}}
\]  

We next prove that \( g(x) > 0 \) for any \( x > 0 \). We start by noting that \( g(x) \) decreases monotonically with \( \alpha \), and hence we can lower bound it by

\[
g(x) \geq \lim_{\alpha \to \infty} g(x) = \lim_{\epsilon \to 0} e^x \cdot E_1(x) - \ln \left( 1 + x^{-1} \left( \Gamma \left( \frac{1}{2} - \frac{2}{\alpha} \right) \right)^{-\frac{1}{2}} \right)
\]

where the last equality used the power series of \( \Gamma(1+z) \), [35], and \( \gamma \) is the Euler-Mascheroni constant (\( \gamma = 0.57721... \)). It is easy to show that \( g(x) > 0 \) \( \forall \ x > 0 \). \( \square \)

This superiority of the fixed transmission power policy over the channel inversion policy was also shown in [7] for the ORD metric. Fig. 1 compares the channel inversion and fixed power transmission strategies. The lower bounds of the fixed transmission power scheme is given by (14), and of the channel inversion scheme is given by (24). As was proven above, the ERD bound for the fixed power scheme is superior to the ERD bound for the channel inversion scheme. The figure shows that this conclusion also holds for the actual ERD and ORD.

3) The Expected ERD Gain from Utilization of Beamforming: From the ratio between (35) and (28) we can find the gain from utilization of \( N_T \) transmit antennas:

\[
\frac{R_{LB}^S(\lambda^S)}{R_{LB}^f(\lambda^f)} = \Gamma \left( 1 + \frac{2}{\alpha} \right) \cdot \left( N_T - 1 - \frac{1}{\alpha} \right)^{\frac{1}{2}}
\]  

Note that for \( N_T \to \infty \), the lower bound (35) scales as \( N_T^{\frac{1}{2}} \). The same scaling was also observed for the ORD [9] for large enough \( N_T \).

Equation (49) anticipates that the expected gain of utilizing 3 and 9 antennas compared to a single antenna is 4.5 and 10.9 respectively. The measured ratio between the ERD utilizing 3 and 9 antennas compared to a single antenna in Figure 4 is 4.3 and 10.8 respectively. Hence, equation (49) is useful for estimating the expected gain as function of the number of transmit antennas.

4) The ERD Gain of Beamforming and Interference Cancellation: Defining \( J_\alpha \triangleq \left( 1 - \frac{2}{\alpha} \right)^{\frac{1}{2}} \),

\[
\Gamma \left( 1 + \frac{2}{\alpha} \right) \Gamma \left( 1 - \frac{2}{\alpha} \right)
\]  

and finding the ratio between (41) and (28), we can obtain the ERD gain from the utilization of \( N_T \times N_R \) antennas.

The gain ratio for 2 \( < \alpha \leq 4 \) and \( N_R \geq 2 \) is:

\[
\frac{R_{LB}^S(\lambda^S)}{R_{LB}^f(\lambda^f)} = J_\alpha \cdot (N_T - 1)^{\frac{1}{2}} \left( N_R - 1 - \frac{\alpha}{4} \right)^{1 - \frac{1}{2}}
\]  

and for \( 4 \leq \alpha < 6 \) and \( N_R \geq 3 \) we get:

\[
\frac{R_{LB}^S(\lambda^S)}{R_{LB}^f(\lambda^f)} = J_\alpha \cdot (N_T - 1)^{\frac{1}{2}} \left( N_R - 2 \right)^{\frac{1}{2}} \left( N_R - \frac{3}{2} \right)^{1 - \frac{1}{2}}
\]  

Note that in the special case where \( N_T = N_R = N \) the ERD scales linearly with \( N \) for large enough \( N \). This scaling was again observed for the ORD metric [8]. Note, however, that [8] only presented asymptotic results for an infinite number of antennas.

Moreover, from (50) and (51) one can observe that for \( \alpha < 4 \) the role of the beamforming and the transmit antennas is more important than the interference cancellation and the receive antennas. For \( \alpha > 4 \) the roles change and the receive antennas has more effect than the transmit antennas.

Fig. 7 depicts the maximal rate density as function of the number of transmit antennas for transmit beamforming with and without interference cancellation at the receiver. In the case of interference cancellation, we assumed the same number of antennas in the receiver and transmitter; i.e., \( N = N_T = N_R \). Equations (41) and (35) were used to plot the lower bound curves for the cases with and without interference cancellation respectively. As shown in subsection V-C3, the curve of the ERD scales as \( N^{\frac{1}{2}} \) for without interference cancellation and linearly with \( N \) for the interference cancellation case.
Moreover, as stated in Corollary 1 the lower bound for the interference cancellation case converges to the ERD for large \( N \).

VI. CONCLUSIONS

In this work we analyzed the ergodic rate density (ERD) of ALOHA WANETs assuming a homogenous PPP distribution of nodes. The ERD is achievable in modern communication systems and was shown to result in a simpler analysis. We presented two novel lower bounds on the ERD. The first bound holds for a general reception strategy and the second bound holds for receivers with spatial interference cancellation.

The usefulness of the two lower bounds was demonstrated by five applications. The first three are single antenna WANETs with different transmission strategies (fixed transmission power, opportunistic ALOHA and channel inversion). The fourth and fifth applications are multiple antenna WANETs applying transmit beamforming, with or without interference cancellation in the receivers. For each application we present closed form expressions of the bounds as function of the system parameters.

These closed form expressions are further used to draw insights on the impact of the system parameters on the ERD. The maximal ERD is shown to grow linearly with the path-loss factor for large values of the path-loss factor. However, for fixed active user density we showed that the ERD is not necessarily a monotonic function of the path-loss factor. We also presented accurate and simple expressions for the maximal-ERD gain as function of the number of antennas with and without interference cancellation. These expressions can be used to study the scaling of the ERD for large number of antennas. In addition, these expressions can also provide the accurate gain for small number of antennas.

The bounds are shown to be tight although fairly simple. These bounds are quite general and can be easily applied to other network models. Future research will adapt these bounds for the analysis of more complicated networks and the joint analysis of additional network layers.

APPENDIX A

PROOF OF THEOREM 1

To prove the theorem we distinguish between two different cases based on the power of the strongest interferer. For this purpose we define the strongest interferer power:

\[
W_{\text{max}} = \max_j W_j
\]

and use the law of total expectation to write the ERD, (4), as

\[
R(\lambda) = \lambda \cdot \Pr(W_{\text{max}} > \delta) \cdot E\left[\log_2\left(1 + \frac{S}{\sigma^2 + I}\right) \mid W_{\text{max}} > \delta\right] + \lambda \cdot \Pr(W_{\text{max}} \leq \delta) \cdot E\left[\log_2\left(1 + \frac{S}{\sigma^2 + I}\right) \mid W_{\text{max}} \leq \delta\right].
\]

(53)

Note that the impact of \( \delta \) on the two terms in (53) is different. For large enough values of \( \delta \) the first term (the case of \( W_{\text{max}} > \delta \)) is very small and can be neglected. On the other hand, the second term (the case of \( W_{\text{max}} \leq \delta \)) becomes small for small values of \( \delta \). Thus, it is important to choose a proper value of \( \delta \) that balances the two terms. Yet, to simplify the bound, we lower bound the first term with 0 and the second term using the Jensen’s inequality, resulting with:

\[
R(\lambda) \geq \lambda \cdot \Pr(W_{\text{max}} \leq \delta) \cdot E\left[\log_2\left(1 + \frac{S}{\sigma^2 + I}\right) \mid W_{\text{max}} \leq \delta\right] \geq \lambda \cdot \Pr(W_{\text{max}} \leq \delta) \cdot E\left[\log_2\left(1 + \frac{S}{\sigma^2 + I}\right) \mid I_{W_{\text{max}}} \leq \delta\right].
\]

(54)

In this case, the Jensen inequality implies that given the average power of a (conditionally) Gaussian interference, the worst interference is the one with constant power. This property was already observed and used in various works (see for example [36], [37]). Note that even though the first term in (53) was lower bounded by 0, it is still important to choose \( \delta \) that will make the bounds as tight as possible.

Let \( \Phi_\delta(s) \) be the characteristic function that corresponds to the conditional distribution of \( I \) given \( W_{\text{max}} \leq \delta \). The characteristic function of an aggregate interference, measured at the middle of a circular guard zone of radius \( A \) within a 2-dimensional PPP with density \( \lambda \) and a fading variable \( K \) is (using [38], [39] and the superposition property, [40], see also [41]):

\[
\Phi(s) = \exp\left(-\lambda E\left[\int_A^{\infty} 1 - e^{-sKr^{-\alpha}} \frac{1}{2\pi r} dr\right]\right).
\]

(55)

where \( A, K \) can be random variables. Therefore, the characteristic function of the conditional aggregate interference, measured at a probe receiver with guard zone of \( \left(\frac{\delta}{\sqrt{1 + \rho(Y)}}, 1\right) \) is:

\[
\Phi_\delta(s) = \exp\left(-\lambda E\left[\int_{\frac{\delta}{\sqrt{1 + \rho(Y)}}}^{\infty} 1 - e^{-sV\cdot \rho(Y)t^{-\alpha}} \frac{1}{2\pi t} dt\right]\right).
\]

(56)

Since the maximum interference contributed by a single interferer is bounded, the first moment of the conditional
aggregate interference exists and can be calculated by:

$$E[I | W_{\text{max}} \leq \delta] = -\frac{d}{ds} \ln (\Phi(s)) \big|_{s=0}$$

$$= 2\pi \lambda E \left[ \int_{\mathbb{R}^2} \left( V \cdot \rho(V) \right) \left( \frac{t}{\alpha + \frac{1}{2}} \right)^{\frac{1}{2}} dt \right]$$

$$= 2\pi \lambda E \left[ \int_{\mathbb{R}^2} \left( V \cdot \rho(V) \right) t^\text{t-\alpha+1} dt \right]$$

$$= \left( \frac{2\pi}{\alpha - 2} \right) E \left[ V \cdot \rho(V) \right] \delta \alpha^{1 - \frac{2}{\alpha}}$$

where the last equality used the definition:

$$L_\alpha \triangleq E \left[ V \cdot \rho(V) \right]$$

The average number of interferers which violate the power threshold condition is:

$$\mathbb{N}(\lambda, \delta) = \sum_k \Pr(W_k \geq \delta)$$

$$= \sum_k \Pr(V_k \rho(Y_k) X_k^\alpha \geq \delta)$$

$$= \sum_k E \left[ \Pr\left( X_k \leq \left( V_k \rho(Y_k) \delta^{-1} \right)^{\frac{1}{2}} | V_k, \rho(Y_k) \right) \right]$$

$$\overset{(a)}{=} \pi \lambda \left( V_k \rho(Y_k) \delta^{-1} \right)^{\frac{1}{2}} E\left[ V \cdot \rho(V) \right]$$

$$\overset{(b)}{=} \pi \lambda \delta \alpha^{1 - \frac{2}{\alpha}} E\left[ V \cdot \rho(V) \right]$$

$$\overset{(c)}{=} \pi \lambda \delta \alpha^{1 - \frac{2}{\alpha}} L_\alpha$$

where equality (a) used the probability of a node to be inside a circle for a distribution of a PPP, equality (b) used the independency between \( r_k, V_k \) and \( \rho(Y_k) \) and equality (c) used (58). For the distribution of a PPP:

$$\Pr(W_{\text{max}} \leq \delta) = e^{-\mathbb{N}(\lambda, \delta)} = e^{-\pi \lambda \delta \alpha^{1 - \frac{2}{\alpha}} L_\alpha}.$$  \(60\)

Substituting (60) and (59) in (54) results in the bound:

$$R(\lambda) \geq R_{\text{LB}}(\lambda, \delta) = \lambda e^{-\pi L_\alpha \lambda \delta^{1 - \frac{2}{\alpha}}} E\left[ \log_2 \left( 1 + \frac{\rho(Y) \cdot Y}{\psi_\alpha} \right) \right].$$  \(61\)

The lower bound in (61) holds for any value of \( \delta \). The best bound can be obtained by maximizing (61) with respect to \( \delta \). However, this approach turns out to be too complicated. Instead we use \( \delta_* \), given in Lemma 2 below, to produce a simpler lower bound. The value of \( \delta_* \) is motivated by the optimization of the bound with respect to both \( \delta \) and \( \lambda \). The resulting bound is \( R_{\text{LB}}(\lambda) = R_{\text{LB}}(\lambda, \delta_*) \) and substituting (63) into (61) completes the proof.

**Lemma 2:** The values \( \delta_* \) and \( \lambda_* \) that solve the joint optimization problem:

$$\delta_*, \lambda_* = \arg \max_{\delta, \lambda} R_{\text{LB}}(\lambda, \delta)$$

satisfy the relation:

$$\delta_* = \left( \frac{\alpha}{\alpha - 2} \right) \pi L_\alpha \lambda_*^{2}$$  \(63\)

**Proof of Lemma 2:** We write the optimization problem in two stages by adding an external optimization on the expectation over the interference:

$$R_{\max} = \max_{\delta, \lambda} R_{\text{LB}}(\lambda, \delta) = \max_c \left( \max_{\delta, \lambda, E[I|c]} R_{\text{LB}}(\lambda, \delta) \right).$$  \(64\)

The proof of the Lemma only needs to consider the internal optimization:

$$(\delta_*, \lambda_*) = \arg \max_{\delta, \lambda} \lambda e^{-\pi L_\alpha \lambda \delta^{1 - \frac{2}{\alpha}}}$$

$$\cdot E\left[ \log_2 \left( 1 + \frac{\rho(Y) \cdot Y}{\psi_\alpha} \right) \right]$$

s.t. \( \frac{2\pi}{\alpha - 2} L_\alpha \lambda^{1 - \frac{2}{\alpha}} = \psi_\alpha \).  \(65\)

which can be simplified to:

$$(\delta_*, \lambda_*) = \arg \max_{\delta, \lambda} \lambda e^{-\pi L_\alpha \lambda \delta^{1 - \frac{2}{\alpha}}} \cdot \psi_\alpha \lambda^{1 - \frac{2}{\alpha}}.$$  \(66\)

Define the following Lagrange function to be maximized:

$$\Lambda(\lambda, \delta) \triangleq \lambda e^{-\pi L_\alpha \lambda \delta^{1 - \frac{2}{\alpha}}} + \psi_\alpha \lambda^{1 - \frac{2}{\alpha}}.$$  \(67\)

where \( \psi \) is a Lagrange multiplier. Any local optimum must satisfy:

$$\frac{\partial \Lambda(\lambda, \delta)}{\partial \lambda} = 0 \Rightarrow e^{-\pi L_\alpha \lambda \delta^{1 - \frac{2}{\alpha}}} (1 - \pi L_\alpha \delta^{1 - \frac{2}{\alpha}}) + \psi_\alpha \lambda^{1 - \frac{2}{\alpha}} = 0$$  \(68\)

and

$$\frac{\partial \Lambda(\lambda, \delta)}{\partial \delta} = 0 \Rightarrow \lambda^{1 - \frac{2}{\alpha}} \left( \frac{2\pi L_\alpha \lambda t}{\alpha} e^{-\pi L_\alpha \lambda \delta^{1 - \frac{2}{\alpha}}} + \delta \left( 1 - \frac{2}{\alpha} \right) \psi \right) = 0.$$  \(69\)

Substituting \( \psi \) from (68) into (69), leads to (63).

**APPENDIX B
PROOF OF THEOREM 2**

Each receiver cancels its M nearest transmitters, and basically creates a geometrical guard-zone with a random radius around itself. Denoting by \( v \) the guard zone radius, its distribution, assuming a density of \( \lambda \), is given by, [42]:

$$f_v(v) = \frac{2(\pi \lambda)^M}{(M - 1)!} v^{2M - 1} e^{-\pi \lambda v^2}.$$  \(70\)

We use (55) to write the following characteristic function of the aggregate interference given a guard zone \( v \):

$$\Phi(s) = \exp \left( -\lambda \int_{v}^{\infty} E \left[ 1 - e^{-sV \rho(V) t^{-\alpha}} \right] 2\pi dt \right)$$  \(71\)
and the expectation of the aggregate interference given \( v \) will be:

\[
E[I|v] = -\frac{d}{ds}\Phi(s)\bigg|_{s=0} = 2\pi \lambda E\left[V \cdot \rho(Y) \int_{v}^{\infty} t^{-\alpha+1} dt\right] = 2\pi \frac{E[V]}{\alpha-2} E[\rho(Y)] \lambda v^{2-\alpha}. \tag{72}
\]

Using (70) leads to:

\[
E[v^{2-\alpha}] = \frac{2(\pi \lambda)^{M}}{(M-1)!} \int_{0}^{\infty} t^{2M-1-\alpha} e^{-\pi \lambda t^{2}} dr = (\pi \lambda)^{2-1} \frac{\Gamma(M+1-\frac{\alpha}{2})}{\Gamma(M)}. \tag{73}
\]

where the second equality used \( M > \frac{\alpha-2}{2} \). Using the chain rule for expectations and substituting (73) into (72) leads to:

\[
E[I] = E_{v} [E[I|v]] = \frac{2\pi}{\alpha-2} E[V] E[\rho(Y)] \lambda \frac{\Gamma(M+1-\frac{\alpha}{2})}{\Gamma(M)}. \tag{74}
\]

From (74), we see that the expectation on the aggregate interference exists and therefore we can use Jensen’s inequality to bound the ERD:

\[
R^{M}(\lambda) = \lambda E\left[\log_{2} \left(1 + \frac{\rho(Y) \cdot Y}{\sigma^{2} + E[I]}\right)\right] \geq \lambda E\left[\log_{2} \left(1 + \frac{\rho(Y) \cdot Y}{\sigma^{2} + E[I]}\right)\right]. \tag{75}
\]

For the last step we use Kershaw’s inequality, [33], [34] for \( 2 < \alpha < 4 \):

\[
\frac{\Gamma(M+1-\frac{\alpha}{2})}{\Gamma(M)} \leq \left(M - \frac{\alpha}{4}\right)^{1-\frac{\alpha}{2}} \tag{76}
\]

and for \( 4 \leq \alpha < 6 \):

\[
\frac{\Gamma(M+1-\frac{\alpha}{2})}{\Gamma(M)} \leq \frac{(M-\frac{1}{2} - \frac{\alpha}{2})^{2-\frac{\alpha}{2}}}{M-1}. \tag{77}
\]

Substituting (76) or (77) into (74) and using (75) completes the proof.

\section*{APPENDIX C

\textbf{PROOF OF COROLLARY 1}}

In order to prove Corollary 1 we have to show that Jensen’s inequality, used in (75) and Kershaw’s inequality, used in (76) and (77), are tight for \( M \to \infty \).

For Jensen’s inequality, we find the variance of the aggregate interference measured at a probe receiver guarded by a geometrical guard-zone of \( v \):

\[
\text{Var}(I|v) = \frac{d^{2}}{ds^{2}} \ln(\Phi(s))\bigg|_{s=0} = 2\pi \lambda E\left[V^{2} \cdot \rho(Y) \int_{v}^{\infty} t^{-2\alpha+1} dt\right] = \frac{2\pi}{2\alpha-2} E[\rho(Y)] \lambda v^{2-2\alpha}. \tag{78}
\]

Using (70) leads to:

\[
E[I^{2-2\alpha}] = \frac{2(\pi \lambda)^{M}}{(M-1)!} \int_{0}^{\infty} t^{2M+1-2\alpha} e^{-\pi \lambda t^{2}} dr = (\pi \lambda)^{\alpha-1} \frac{\Gamma(M+1-\alpha)}{\Gamma(M)}. \tag{79}
\]

for \( M > \alpha - 1 \). We will also use

\[
E[I^{4-2\alpha}] = (\pi \lambda)^{\alpha-2} \frac{\Gamma(M+2-\alpha)}{\Gamma(M)} \tag{80}
\]

for \( M > \alpha - 2 \). Using the low of total variance leads to

\[
\text{Var}(I) = E[I^{2}] Var(I|v) + Var(E[I|v]) = \frac{2(\pi \lambda)^{\alpha}}{2\alpha-2} E[V^{2}] E[\rho(Y)] \frac{\Gamma(M+1-\alpha)}{\Gamma(M)} + \left(\frac{2}{\alpha-2} E[V] E[\rho(Y)]\right)^{2} \left(\frac{\Gamma(M+2-\alpha)}{\Gamma(M)}\right)^{2}. \tag{81}
\]

where the last equality used (72), (74), (80) and the substitution of (79) into (78).

Directly from Stirling’s formula, [43], we obtain the following Gamma functions asymptotic ratio:

\[
\lim_{M \to \infty} \frac{\Gamma(M+a)}{\Gamma(M+b)} \cdot M^{b-a} = 1. \tag{82}
\]

Thus, \( E[I] \) scales as \( M^{1-\frac{\alpha}{2}} \) and \( \text{Var}(I) \) is scales as \( M^{1-\alpha} \). Hence,

\[
\lim_{M \to \infty} \text{Var}(M^{\frac{\alpha}{2}-1} \cdot I) = 0 \tag{83}
\]

and we can state

\[
\lim_{M \to \infty} M^{\frac{\alpha}{2}-1} \cdot I = \frac{2\pi}{\alpha-2} E[V] E[\rho(Y)] \lambda^{\frac{\alpha}{2}}. \tag{84}
\]

Therefore, \( M^{\frac{\alpha}{2}-1} \cdot I \) converges to a deterministic constant, and Jensen’s inequality is tight.

The Gamma functions asymptotic ratio, (82), also implies that the two sides of Kershaw’s inequality used in (76) and (77) converges to equality for \( M \to \infty \). \qed

\section*{REFERENCES}


Chapter 4

Upper Bound on the Ergodic Rate Density of Poisson Wireless Ad-hoc Networks

Abstract—We present a novel upper bound on the Ergodic Rate Density (ERD) of ALOHA wireless ad-hoc networks. Our analysis uses a proper model of the physical layer together with an abstraction of higher communication layers. The novel bound is very general and supports various system models including for example, beamforming, spatial multiplexing, different fading models and different power control schemes. We also derive a closed form expression for the maximal gap between the novel bound and a known lower bound on the ERD. This maximal gap holds for any network that operates below the optimal density. This expression is simple to evaluate and only depends on the path loss factor. For example, for a path loss factor of $\alpha = 3$ the novel upper bound is proved to be at most 31% higher than the lower bound (and hence also from the actual ERD). The usefulness and the generality of the novel bound is demonstrated by applications in multiple-antenna schemes. In particular, we study the optimization of the number of transmitted spatial streams in a MIMO network and derive the scaling of the ERD as the number of antennas grows. The results are further demonstrated using extensive simulations.

I. INTRODUCTION

Wireless Ad-hoc Networks (WANETs) offer simplicity and flexibility that make them suitable for many practical applications. These networks rely on decentralized channel access protocols (e.g., ALOHA [1] and Carrier Sense Multiple Access (CSMA) [2, 3]). Thus, WANETs can provide reliable communication without the need for any infrastructure.

The following work focuses on the Ergodic Rate Density (ERD) metric. The ERD metric considers the communication rate of each pair as the mutual information between the transmitted signal and the received signal given the interferers’ activity.

From the practical point of view, it is most interesting to predict and optimize the performance of a specific WANET. Yet, such works are hard to generalize, and usually do not give much insight on the operation of WANETs. Thus, many works have turned to analyze random networks, using various distributions.

The simpler and the most popular model for the position of nodes in random WANETs, is the homogeneous Poisson Point Process (PPP) [4] model. In this model the number of users in each finite area has a Poisson distribution, and their locations are uniformly distributed over the area. Obviously, such a random model cannot predict the performance of any specific network. Yet, this intuitive model, which can be seen as an extension of the uniform distribution into an infinite plane, offers a simple description of an unknown network, without the need to specify the network characterization. In fact, exact definition of the PPP model requires only a single parameter - the density of the nodes on the plane. Thus, analysis of PPP distributed WANETs have attracted much attention (e.g., [5]–[7]) under the hope that the insights that come from such analysis will hold also in practical networks.

For the analysis of random WANETs, we follow the approach that uses exact modeling of the physical layer, together with a simplified abstraction of the other networking layers [4], [7]–[9]. In this approach, performance is typically evaluated using the network Area Spectral Efficiency (ASE), which is defined as the density of communicating pairs multiplied by their communication rates. The Transmission Capacity (TC) metric [10] is defined as the maximum achievable ASE given an outage constraint where all users use a fixed transmission rate. The ERD metric [11], [12] was shown to be higher than the TC metric for ALOHA WANETs (see for example [9], [13]).

The ERD can be asymptotically approached in ALOHA WANETs if the transmission scheme can achieve a sufficiently high diversity order. Some of the relevant transmission schemes were detailed in [9] including: a time diversity scheme in which each code word is spread over multiple ALOHA packets (the diversity order of this scheme was studied in [14]), a lower delay scheme that utilizes time diversity through an incremental-redundancy hybrid automatic repeat request (IR-HARQ) protocol [15]–[17] and a frequency diversity scheme, which spreads each message over multiple carriers (see also [12]).

The study of the ERD in WANETs has become somewhat simpler with the recent introduction of a general lower bound [9] which is applicable to any pre/post processing, fading distribution and power allocation scheme. The tightness of the bound was studied through simulations, and it was shown that the bound gives a good description of the behavior of the true ERD. However, [9] did not present an upper bound, and hence, the tightness of the bound was not evaluated analytically.

So far, the only upper bound on the ERD of ALOHA WANETs was presented by Stamatiou et al. [12] for the case where the nodes are equipped with multiple antennas and utilize a frequency hopping protocol. However, the model of Stamatiou et al. assumes constant transmission power and an absence of information in the receivers on the realization of the interfering channels. Both of these assumptions limit the applicability of the bound. The lack of channel state information (CSI) at the receivers characterizes specific networks. However, performance without receiver CSI is generally lower, [18], and the upper bound in Stamatiou et al. does not apply to more general networks. Moreover, the assumption of constant transmission power does not allow the study of power
allocation schemes that may lead to higher performance.

In this work we generalize the bound presented in Stamatiu et al., and present a novel upper bound on the ERD of ALOHA WANETs that is applicable to the more general system model, presented in [9]. Thus, the novel bound is applicable to general power control schemes and general distributions of fading channels. Comparing the novel upper bound to the known lower bound of [9] enables us to present a closed form expression that guarantees the maximum gap between the two bounds. For example, for a path loss factor in the range of 2.5 to 4, the difference between the bounds is shown to be at most in the range 18% to 47% respectively.

The generality of the upper bound is further used to obtain closed form expressions for the performance of WANETs utilizing Multiple-In-Multiple-Out (MIMO) antenna techniques.

So far, the performance of random MIMO WANETs have been studied mostly by the use of the TC performance metric (see for example [7] for various antenna diversity techniques). As stated above, in this work we focus on the ERD metric. In addition to its higher performance, the ERD metric also results in bounds that are simpler than the equivalent bounds on the TC. Furthermore, a typical TC framework includes an assumption of a small outage constraint [10], whereas the maximum spectral efficiency is typically achieved with large outage probabilities. The ERD upper bound presented here is valid for any user density and system parameters, and hence provides a simple and efficient tool for network optimizations. In the case of spatial multiplexing, the products of these optimizations are the optimum network density and the optimum number of streams.

Several works have investigated the tradeoff between spatial multiplexing and beamforming in MIMO WANETs. When the receivers perform interference cancellation of the undesired transmissions the optimum number of streams was shown to be one [8]. On the other hand, when the interference was considered as noise, spatial multiplexing was shown to have a potential gain [19]. In particular, increasing the number of streams was shown to be effective when the interference is limited (i.e., large path loss factor, large number of antennas or small density of users). The closed form expressions of the novel upper bound show that the insights in [19] are also valid for the ERD metric. Moreover, these expressions can be used to optimize the ERD for any active user density. We also show that the scaling gain of eigen-beamforming as a function of the number of transmit and receive antennas can be easily deduced from the bound for optimized density WANET and Rayleigh fading.

The rest of this paper is organized as follows: Section II describes the system model. Section III introduces the novel upper bound and an analytical evaluation of the tightness of the bound. Section IV introduces two applications of the novel upper bound for WANETs utilizing multiple antennas and Section V presents our concluding remarks.

A note on notation: For a matrix $A$ the notation $A^\dagger$ denotes the conjugate transpose and for a vector $v$ the notation $\|v\|^2 = v^\dagger v$ denotes its square Frobenius norm.

II. SYSTEM MODEL

A. Slotted ALOHA

We assume a decentralized wireless ad-hoc network utilizing an ALOHA protocol. For simplicity of presentation, we state the results only on the slotted variant of ALOHA (e.g., [4]). The applicability of the results also to the unslotted variant of ALOHA is discussed in Subsection II-B. For the position of nodes we use the popular Poisson bipolar network model [20], [21]. In this model the transmitters are distributed as a two dimensional PPP on the plane with a density of $\lambda$. We assume that each receiver is located within a fixed distance, $d$, from its pair transmitter.

We assume that each node is equipped with $N$ transmit antennas and $M$ receive antennas. The channel matrix from transmitter $j$ to the receiver $i$ is denoted by $H_{ij} \in (M \times N)$ with i.i.d. distributed entries. Each transmitter is assumed to perform spatial multiplexing of $K$ streams [22] with $1 \leq K \leq \min(M,N)$.

The received signal at receiver $i$ is given by:

$$r_i = \sum_j \sqrt{\rho_j} x_{ij}^\dagger H_{ij} z_j + n_i$$

where $\rho_j$ is a scalar which denotes the transmission-power of the $j$-th transmitter and can depend only on the desired channel. $x_{ij}$ is a scalar which denotes the distance between the $j$-th transmitter and the $i$-th receiver respectively, $z_j$ is the transmitted signal from transmitter $j$ and $n_i$ is the thermal noise. The path-loss factor is denoted by $\alpha$ (and the analysis is limited to $\alpha > 2$, which is required to bound the received energy). We assume that the network is operating at the interference limited regime, and therefore, in the remainder of the analysis we neglect the contribution of the thermal noise (setting $n_i = 0$). Note that in this case normalizing the transmission power by a constant will not affect the final result.

We assume that the $i$-th transmitter has perfect channel knowledge of $H_{ii}$, the channel matrix to the desired receiver, but no knowledge of any other channel matrix in the network. The Singular Value Decomposition (SVD) of the desired channel is given by $H_{ii} = U_i D_i W_i^\dagger$, where $U_i$ and $W_i$ are unitary matrices, and $D_i = \text{diag} \left( [\gamma_i, \ldots, \gamma_i, \min(M,N)] \right)$ is a diagonal matrix with the singular values of $H_{ii}$ on its diagonal. Without loss of generality, we will assume throughout that the singular values are ordered so that $\gamma_i,1 \geq \gamma_i,2 \geq \cdots \geq \gamma_i,\min(M,N)$. The optimal precoding vector of the $k$-th stream at the $i$-th transmitter is given by the $k$-th column of $W_i$, denoted by $w_{i,k}$. The precoded signal of the $i$-th transmitter is therefore given by:

$$z_i = \sum_{k=1}^{K} w_{i,k} z_{i,k}$$

where $z_{i,k}$ denotes the data symbol of the $k$-th data stream for the $i$-th pair.

The receiver optimal decoding vector for the $k$-th stream is given by the $k$-th row of $U_i^\dagger$, denoted by $u_{i,k}^\dagger$, and hence the

\[1\]The bound can be trivially extended to any desired distribution of the distance to the desired transmitter.
i-th receiver post-processed signal for the k-th stream is given by:

$$\tilde{r}_{i,k} = u_{i,k}^j r_i.$$ \hfill (3)

Using the shift invariance property, [23], we analyze the performance of the network using a probe receiver. Without loss of generality, we assume that the probe receiver is located at the origin. For notational simplicity we drop the probe receiver index from all relevant variables.

The k-th stream of the desired user is received at the probe receiver with a power of:

$$S_k = \rho_0 d^{-\alpha} \left\| u_k^j [H_0 W] \right\|^2 = \rho_0 d^{-\alpha} \nu_k.$$ \hfill (4)

where $\nu_k$ is the square of the absolute value of the k-th largest singular value of $H_0$. Without loss of generality we assume that $d = 1$.

The interference power contributed by transmitter $j$ for the detection of the k-th spatial stream is given by:

$$T_{j,k} = \left\| \sqrt{\rho_j} X_j^{-\alpha} u_k^j [H W] \right\|^2 = \rho_j X_j^{-\alpha} \eta_{j,k}$$ \hfill (5)

where the second line used the definition $\eta_{j,k} \triangleq \left\| u_k^j [H W] \right\|^2$. As we assume that all channel matrices are statistically independent, the fading variables $\eta_{j,k}$ are statistically independent, and also statistically independent of all distance and transmission-power variables. Moreover, all fading variables that describe the k-th stream, $\eta_{j,k}$, have the same distribution.

The power of the aggregate interference, measured at the probe receiver in the k-th stream detector, is given by:

$$I_k = \sum_j T_{j,k} = \sum_j \rho_j X_j^{-\alpha} \eta_{j,k}.$$ \hfill (6)

The ergodic rate density (ERD) of the k-th spatial-stream of a network with an active user density of $\lambda$ is given by [9]:

$$R_k (\lambda) = \lambda \cdot E \left[ \log_2 \left( 1 + \frac{S_k}{I_k} \right) \right]$$ \hfill (7)

In order to generalize our analysis we define $Y$ as a Random Variable (R.V.) that represents the received power at the probe receiver from its paired transmitter (e.g., for the reception of the k-th stream, $Y = S_k$) and $\{V_j\}$ as the R.V.s that represent the power of the channel between the receiver and the j-th transmitter, and define the generic ERD:

$$R_{f_Y,f_Y,f_P} (\lambda) = \lambda \cdot E \left[ \log_2 \left( 1 + \frac{Y}{\sum_j \rho_j X_j^{-\alpha} V_j} \right) \right].$$ \hfill (8)

Note that in our model, the transmitted power of each node depends solely on the gain to its paired receiver. Thus, in (8), $X_j$ and $V_j$ are statistically independent of the desired channel of the j-th transmitter and hence also independent of $\rho_j$. We use the notations $V$ and $\rho$ only when we need an arbitrary representative of the corresponding R.V. family. Thus, $f_Y$ and $f_P$ are the Probability Distribution Functions (PDFs) of the R.V.s $V_j$ and $Y$ respectively, and $f_{\rho}$ is the PDF of the R.V.s $\rho_j$.

Using the generic ERD definition of (8) we can now write the expression for the ERD of the probe pair, utilizing $K$ spatial streams as:

$$R(\lambda) = \sum_{k=1}^K R_k (\lambda) = \sum_{k=1}^K R_{f_Y,f_Y,f_P} (\lambda).$$ \hfill (9)

Note that in the following we sometimes drop the subscript notations $f_Y, f_Y, f_P$ when the relevant distributions are easily understood from the context.

A useful lower bound on the ERD, which is helpful for the analysis herein, was originally presented in [9]. This lower bound on the ERD is given by:

$$R_{f_Y,f_Y,f_P} (\lambda) \geq R_{LB,f_Y,f_Y,f_P} (\lambda)$$ \hfill (10)

where

$$R_{LB,f_Y,f_Y,f_P} (\lambda) = \lambda \cdot E \left[ \log_2 \left( 1 + \frac{Y}{\sum_j \rho_j X_j^{-\alpha} V_j} \right) \right].$$ \hfill (11)

and

$$C_{f_Y,f_Y,f_P} \triangleq \frac{2}{\alpha (\alpha - 2) \pi^2} \left( \frac{\pi \alpha E [V^2]}{E \left[ \rho^2 \right]} \right)^{\frac{1}{2}}.$$ \hfill (12)

B. Applicability to unslotted ALOHA

In the unslotted ALOHA protocol, the participating transmitters deliver their packets in an asynchronous manner. This results in variations of the aggregate interference power during a packet reception. Thus, instead of doing the analysis by slots, we need to analyze the mutual information per symbol. In this case, Equations (1)-(6) describe the model for a single packet reception. Without loss of generality we assume that all results are applicable for an unslotted ALOHA network if each receiver is capable of estimating the variable interference power.
III. PERFORMANCE ANALYSIS

In the following section we present two theorems. The first theorem formulates a novel and general upper bound on the ERD. The second theorem provides information on the tightness of the bound, by bounding the ratio between the lower bound of [9] and the novel upper bound.

Theorem 1: An upper bound on the ERD of a network with an active user density of \( \lambda \) is:

\[
R_{fV,fV,fP}(\lambda) \leq R_{UB,fV,fV,fP}(\lambda)
\]

where

\[
R_{UB,fV,fV,fP}(\lambda) = \lambda \cdot E \left[ \log_2 \left( 1 + \frac{Y}{\Lambda_\alpha \cdot \lambda^2} \right) \right]
\]

and \( \Lambda_{\alpha,fV,fP} \) is defined in (15).

\[
\Lambda_{\alpha,fV,fP} = \frac{\pi E \left[ V^2 \right] E \left[ \rho^2 \right] \Gamma \left( 1 - \frac{2}{\alpha} \right) \left( 1 + \frac{2}{\alpha} \right)}{\Gamma \left( 1 + \frac{2}{\alpha} \right)}.
\]

Note that both the upper and lower bounds, (14) and (11), have a similar formulation. We next define:

\[
\Omega_\alpha \triangleq C_{\alpha,fV,fP} \cdot \Lambda_{\alpha,fV,fP}^{\frac{2}{\alpha}}
\]

and scale the density of the upper bound by \( \Omega_\alpha^{-1} \) compared to the density of the lower bound. Hence, the relation between the upper bound and the lower bound can be expressed by:

\[
R_{LB,fV,fV,fP}(\lambda) = e^{\frac{2}{\alpha}} \cdot \Omega_\alpha \cdot R_{UB,fV,fV,fP}(\lambda \cdot \Omega_\alpha^{-1}).
\]

Recalling that \( R_{UB}(0) = 0 \) and \( R_{UB}(\lambda) > 0 \) for any \( \lambda > 0 \), there cannot be any local minimum of \( R_{UB}(\lambda) \) in the range \( 0 < \lambda < \lambda_{UB}^* \). The lower bound function exhibits the same behavior, and we also define:

\[
\lambda_{LB}^* = \min \left\{ \lambda : \frac{\partial R_{LB}(\lambda)}{\partial \lambda} = 0, \frac{\partial^2 R_{LB}(\lambda)}{\partial \lambda^2} < 0 \right\}.
\]

Using (17) leads to the simple relation:

\[
\lambda_{LB}^* = \Omega_\alpha \cdot \lambda_{UB}^*.
\]

Theorem 2: For an active user density of \( \lambda \leq \lambda_{LB}^* \), the ratio between the upper bound and the ERD is bounded by:

\[
\frac{R_{fV,fV,fP}(\lambda)}{R_{UB,fV,fV,fP}(\lambda)} \leq \frac{R_{fV,fV,fP}(\lambda)}{R_{fV,fV,fP}(\lambda)} \geq e^{\frac{2}{\alpha}} \cdot \Omega_\alpha
\]

where \( \Omega_\alpha \) is defined in (18).

Proof of Theorem 1: See Appendix A.

Proof of Theorem 2: See Appendix B.

To demonstrate the tightness of the upper and lower bounds we present a comparison of the derived analytical expressions and the actual ERD which is evaluated through simulations. The ERD in the simulations was evaluated using a Monte-Carlo network simulator averaging 10^7 network realizations. The channels between each transmit and receive antenna are distributed as Rayleigh fading.

\footnote{Bounds on the ratio of two functions with a similar formulation were presented in [18] for the specific case of Rayleigh fading without the assumption of a single extremum.}
Fig. 1. Rate density as a function of active user density for $\alpha = 3$ and Rayleigh fading channels.

Fig. 2. Maximal rate density as a function of the path loss factor for Rayleigh fading channels.

scattering environment which results in a Rayleigh fading channel between each transmit and receive antenna in the network. The channels are assumed to be normalized and the channel matrix $H$ distributes as a central complex Wishart matrix [22], [27] with $CN(0, 1)$ entries$^3$.

A. Single Stream Beamforming

We next analyze the case of transmit beamforming with $N$ transmit antennas and a single receive antenna, i.e., $M = 1$. We also assume the use of a simple channel inversion power control strategy (i.e., we choose $\rho$ to be $\rho = \nu^{-1}_k$ so that $Y = 1$) which is performed independently among all pairs of transmitters and receivers.

The preprocessing of beamforming over $N$ antennas results in a Chi-square desired channel distribution, with $2N$ degrees of freedom, $2\nu_k \sim \chi_{2N}^2$, which leads to:

$$E \left[ \rho^\frac{\alpha}{2} \right] = \frac{\Gamma \left( N - \frac{\alpha}{2} \right)}{\Gamma(N)}.$$  \hspace{1cm} (25)

This preprocessing does not change the interference channel statistics, $V$. In this case $V = \eta_k$ has an exponential distribution and hence, the expectation over $V$ in (15) is:

$$E \left[ V^\frac{\alpha}{2} \right] = \Gamma \left( 1 + \frac{2}{\alpha} \right).$$  \hspace{1cm} (26)

Substituting (26) and (25) into Theorem 1 leads to:

$$R^{B\text{UB}}_{\text{UB}}(\lambda) = \lambda \cdot \log_2 \left( 1 + \frac{1}{\Lambda^B_{\alpha}} \right)$$ \hspace{1cm} (27)

where we applied the inverse power control scheme in the numerator, i.e., $Y = 1$, $B$ stands for beamforming and

$$\Lambda^B_{\alpha} \triangleq \left( \pi \frac{\Gamma \left( N - \frac{\alpha}{2} \right)}{\Gamma(1 + \frac{\alpha}{2})} \right)^{\frac{2}{\alpha}} \frac{\Gamma \left( 1 + \frac{2}{\alpha} \right) \Gamma \left( 1 + \frac{\alpha}{2} \right)}{\pi \Gamma \left( 1 + \frac{\alpha}{2} \right) \Gamma(N)}.$$  \hspace{1cm} (28)

This upper bound has the same form as the lower bound on the ERD for the case of transmit beamforming in [9],

$^3$Recall that we assume to be working in the interference limited regime. Hence, the average transmission power has no impact on network performance.
except for parameter $\Lambda_\alpha^K$. Thus, the closed form expression for optimal user density can be easily adapted from the results in [9], and is given by:

$$\lambda^B = \arg \max_\lambda R^B_{\text{UB}}(\lambda) = \left(\Lambda_\alpha^K (e^{\Xi_{\alpha}}-1)\right)^{-\frac{1}{\alpha}}$$  \hspace{1cm} (29)

where $\Xi_{\alpha}$ is defined in (23). The optimum upper bound can be written as:

$$R^B_{\text{UB}}(\lambda^B) = \frac{\lambda^B \cdot \Xi_{\alpha}}{\ln(2)}. $$  \hspace{1cm} (30)

The ergodic rate density for the case of transmitter beamforming and channel inversion power control with various number of antennas is illustrated in Fig. 4. The figure depicts the upper bound, (27), the ERD, and the lower bound, [9, eq. 30] as a function of the active user density for $M = 1$, $N = 1, 3, 9$, and a path loss factor of $\alpha = 3$. The analytical maximum rate density of the upper bound (30) is 0.06, 0.27 and 0.65 whereas the measured maximum ERD is 0.053, 0.24 and 0.57 for $N = 1, 3, 9$ respectively. In all three cases the ERD is at most 13% below the upper bound which indicates the tightness of the upper bound and its usefulness for optimal ERD evaluation.

### B. Spatial Multiplexing

In this subsection we analyze the performance of WANETs that utilize spatial multiplexing. We assume that each pair is equipped with an equal number of transmit and receive antennas; i.e., $N = M$ and that each pair delivers its information using a constant number of spatial streams, denoted by $K$, where $1 \leq K \leq M$. We further assume a fixed transmission power strategy, i.e., $p = 1$, and that each pair delivers its data over the first (largest) $K$ singular values.

The ERD for the spatial multiplexing case is given in (9). We next consider the bounding of the ERD of each spatial stream using Theorem 1.

For a transmission of $K$ spatial streams the interference power contributed by each interferer is distributed as a Chi-square with $2K$ degrees of freedom and a mean of $K$; i.e., $2V \sim \chi^2_{2K}$ which results in:

$$E \left( [\eta_k]^\frac{2}{\alpha} \right) = E \left( V^\frac{2}{\alpha} \right) = \frac{\Gamma \left( K + \frac{2}{\alpha} \right)}{\Gamma \left( 1 + \frac{2}{\alpha} \right) \Gamma (K)}. $$  \hspace{1cm} (31)

On the other hand, the generic variable for the desired power, denoted by $Y$ in Theorem 1, is now be substituted (for the $k$-th spatial stream) by $\nu_k$, which is the power of the $k$-th singular value of the channel matrix. Substituting (31) and using Theorem 1 results in the following upper bound for the spatial multiplexing case,

$$R^S_{\text{UB}}(\lambda, K) = \lambda \cdot \sum_{k=1}^K E \left[ \log_2 \left( 1 + \frac{\nu_k}{\Lambda_{\alpha,K}^{S,K}}/\lambda^\frac{2}{\alpha} \right) \right] $$  \hspace{1cm} (32)

where $S$ stands for spatial multiplexing, and

$$\Lambda_{\alpha,K}^{S,K} = \left( \frac{\pi \Gamma \left( K + \frac{2}{\alpha} \right) \Gamma \left( 1 + \frac{2}{\alpha} \right) \Gamma (K)}{\Gamma \left( 1 + \frac{2}{\alpha} \right) \Gamma (K)} \right)^{\frac{1}{\alpha}}.$$  \hspace{1cm} (33)

Note that (33) can be further simplified by using Kershaw’s inequality, [28], [29]:

$$\frac{\Gamma \left( K + \frac{2}{\alpha} \right)}{\Gamma (K)} \geq \left( K - \frac{1}{2} + \frac{1}{\alpha} \right)^{\frac{1}{\alpha}} $$  \hspace{1cm} (34)

which is very tight for all values of $\alpha > 2$ and $K \geq 1$.

The number of streams, $K$, affects the upper bound in three different ways in (32). The sum outside the expectation increases the ERD as the number of streams increases. The second effect is on the distribution of the $k$-th stream power, which is obviously less preferable than the power of previous streams (due to the assumption that the singular values are ordered in decreasing order). Thus, the expected rate of the $k$-th stream is lower than the rate of previous streams. These two effects are similar in nature to the behavior of MIMO in the single user scenario. But in the WANET case, there is also a third effect, through the effect of the number of streams on the interference. This effect is seen through the $\Lambda_{\alpha,K}^{S,K}$ term in
the denominator of (32) as $K$ increases (considering (34), for large $K$, $\Lambda_s^{S,K}$ is approximately linear with $K$). This increase in the interference term has a negative impact on all streams (and not only on the last stream added). Thus, the use of a large number of spatial streams is less favorable than in the MIMO case, even if the transmission power is high. Hence, finding the optimal number of streams in (32) is not trivial.

It is also interesting to compare the increase in the number of spatial streams per user to the increase in the user density by the same ratio in (32). In both approaches the overall number of streams in the network remain constant. However increasing $K$ will approximately increase the interference linearly with $K$ whereas increasing the active user density will increase the interference by a factor of $\Lambda_s^2$. This can be explained by two differences between the two approaches. The first difference is that the receiver achieves complete interference cancellation between the streams of the desired user. Thus, increasing $K$ has a smaller effect on the received interference than the equivalent increase in the number of nodes. Furthermore, increasing the node density also increases the probability of having a nearby interferer whereas increasing the number of streams only affects the distribution of the power contributed by each interferer. Thus, in terms of interference, it is always more beneficial to increase $K$ than to increase the node density by the same ratio.

On the other hand, if the interference term, $\Lambda_s^{S,K} \cdot \lambda^2$, is very large, the upper bound (32) can be approximated by:

$$R^S_{UB}(\lambda, K) \simeq \frac{\lambda^{1-\frac{2}{\alpha}} \Gamma \left(1 + \frac{\alpha}{2}\right)}{\pi \Gamma \left(1 - \frac{\alpha}{2}\right)} \frac{1}{K - \frac{1}{2}} \sum_{k=1}^{K} E[\nu_k] \tag{35}$$

where we used $\log(1 + x) \simeq x$ and (34). Recalling that $\alpha > 2$, the expression in (35) is clearly optimized by the eigen-beamforming scheme, i.e., $K = 1$.

Optimizing the network parameters of (32) requires a two dimensional optimization. This optimization can be performed in two steps. The first step optimizes the density of active users for each number of streams, $K$, i.e.:

$$R^S_{UB} (\lambda_s^{S,K}, K) = \max_{\lambda} R^S_{UB}(\lambda, K), \ K \in \{1, 2, \ldots, M\}. \tag{36}$$

The second step optimizes the number of streams:

$$R^S_{UB} (\lambda_s^{S,K}, K) = \max_{K \in \{1, 2, \ldots, M\}} R^S_{UB} (\lambda_s^{S,K}, K). \tag{37}$$

Fig. 5 depicts the upper bound (32) as a function of the active user density for $N = M = 6$, $\alpha = 4$, Rayleigh fading channels, and $K = 1, 2, 3, 4, 5, 6$. The figure shows that for a low density of active users ($\lambda < 0.03$) the optimal number of streams is equal to the maximum available; i.e., $K = 6$. When the density of nodes grows, the optimal number of streams decreases and for a high density of active users ($\lambda > 0.75$) eigen-beamforming ($K = 1$) is the optimal selection. Note also that the ERD has a single maximum for each selection of $K$. Further insights on the curves’ maxima can be derived from Fig. 6.

Fig. 6 depicts the maximum rate density of the upper bound (presented in (36)), the ERD and the rate density of the lower bound (integration of (11) and (9)) as a function of the optimal active user density for $N = M = 6$, and $\alpha = 4$. The figure only shows the point that represents the maximum of each curve, for $K = 1, 2, 3, 4, 5, 6$. For each $K$ the location of the point is determined according to the maximum ERD achieved by this number of streams (y-axis) and the active user density that achieves this maximum (x-axis). Note that the optimal user density $\lambda_s^{S,K}$ is a decreasing function of the number of spatial streams, $K$.

As can be seen, the curve of both bounds exhibits the same behavior as the actual ERD. But the upper bound gives better predictions for the optimal user density for any given number of streams. In addition, all curves are maximized for $K^* = 3$. Thus, the optimal active user density (defined by (37)) is $\lambda_s^* = 0.25$, the maximal upper bound on the ERD is $R^S_{UB} (\lambda_s^*, K^*) = 1.66$ bit/sec/m$^2$ and the maximal ERD is $1.22$ bit/sec/m$^2$. Note that the ratio between the maximum of the lower bound and the maximum of the upper bound is exactly 54% as was anticipated by Theorem 2.

Unlike the case of single user MIMO, the additional gain from spatial multiplexing in density-optimized WANETs seems to be quite limited compared to the eigen-beamforming scheme ($K = 1$). Capacity analysis in single user MIMO shows that the spectral efficiency increases monotonically with the number of spatial streams, and that the capacity gain approaches $M$ for high enough SNR. In contrast, in the WANET MIMO case, the ERD is not monotonic with the number of spatial streams. In the example in Fig. 6, plotted for the high SNR regime and density-optimized WANETs, the gain from the selection of the optimal number of streams compared to the eigen-beamforming scheme is only 20%. It is important to emphasize that the limited gain of spatial multiplexing does not mean that the gain of multiple antennas is limited. Rather, it shows that in a density optimized WANET, even a single stream WANET gains significantly from the use of multiple antennas.

The upper bound can also be used to analyze the asymptotic behavior of MIMO WANETs as the number of antennas...
Fig. 6. The maximum rate density of the upper bound, the ERD and the lower bound as a function of the optimal active user density for MIMO $6 \times 6$, $\alpha = 4$ and $K = 1, 2, 3, 4, 5, 6$ spatial streams.

grows, i.e., $M \to \infty$. We first analyze the density optimized eigen-beamforming case. For Rayleigh fading, and asymptotic number of antennas the distribution of the largest singular value, $\nu_1$, converges to a Tracy-Widom distribution \cite{30} with an expected value of $\mathbb{E}[\nu_1] = \left(\sqrt{M - 1} + \sqrt{M}\right)^2$ and a standard deviation of

$$\sigma_{\nu_1} = \left(\sqrt{M - 1} + \sqrt{M}\right) \cdot \left(1/\sqrt{M - 1} + 1/\sqrt{M}\right)^{1/3}.$$  

Thus, the normalized largest singular value, $\nu_1/M$, satisfies

$$\lim_{M \to \infty} \frac{\nu_1}{M} = 4 \quad \text{and} \quad \lim_{M \to \infty} \frac{\nu_1}{M} = 0,$$

and hence it converges in mean square to its expected value, i.e., $\lim_{M \to \infty} \frac{\nu_1}{M} \xrightarrow{M.S.} 4$.

Substituting the expected value, 4, into (32) and using the change of variables $\lambda' = \lambda M^{-\frac{1}{2}}$ results in

$$\lim_{M \to \infty} R_{\text{UB}}^S (\lambda', 1) = \lambda' \cdot \log_2 \left(1 + \frac{4}{\Lambda_{\alpha} \cdot (\lambda')^{\frac{1}{2}}} \right) \quad (38)$$

where we also relied on the bounded derivative of $\log(1 + x)$ for positive values of $x$. Thus, the upper bound on the density optimized ERD for single stream transmission scales as $M^\frac{1}{2}$, and it can be approximated as

$$R_{\text{UB}}^S (\lambda', 1) \simeq M^\frac{1}{2} \cdot \max_{\lambda' \geq 0} \lambda' \cdot \log_2 \left(1 + \frac{4}{\Lambda_{\alpha} \cdot (\lambda')^{\frac{1}{2}}} \right). \quad (39)$$

This result matches the TC scaling derived for the case of eigen-beamforming in \cite{7}. Note that \cite{31} showed a linear scaling of the SIR with the number of antennas. This linear scaling can be translated into linear scaling of the ERD, but only under a strong interference assumption. This assumption does not hold when the number of antennas is very large, and hence the actual scaling of the ERD is only as $M^\frac{1}{2}$.

In order to analyze the case of multiple stream transmission we allow $K$ to increase with $M$, but keep the ratio between the number of streams and the number of antennas as constant, denoted by $\beta \triangleq K/M$. Define the empirical Cumulative Density Function (CDF) of the normalized stream power, $F_{\psi} (\lambda) = \frac{1}{M} \sum_{i=1}^{\infty} 1 \left(\frac{\nu_i}{M} < \lambda\right)$, where $1(\text{condition})$ is the indicator function which equals 1 if the condition is satisfied and zero otherwise. For an asymptotic number of antennas and Rayleigh fading, the empirical CDF converges, and the corresponding PDF is given by \cite{32}:

$$\lim_{M \to \infty} f_{\psi} (\lambda) = \frac{1}{2\pi} \sqrt{\frac{4 - \lambda}{\lambda}} \quad \text{(40)}$$

where $0 < \lambda < 4$. Defining $\psi(\beta) \triangleq \lim_{M \to \infty} \frac{\nu_K}{M}$ leads to the relation:

$$\frac{1}{2\pi} \int_{\psi(\beta)}^{4} \sqrt{\frac{4 - x}{x}} \, dx = \beta. \quad \text{(41)}$$

Thus, for an asymptotic number of antennas the summation in the upper bound, (32), converges to:

$$\lim_{M \to \infty} R_{\text{UB}}^S (M, K, \alpha) = \lim_{M \to \infty} \frac{1}{M} \lambda \int_{\psi(\beta)}^{4} \log_2 \left(1 + \frac{M \cdot x}{\Lambda_{\alpha} \cdot \lambda^\frac{1}{2}} \right) \cdot \left(M \cdot f_{\psi} (\lambda) \right) \, dx$$

$$= \frac{\lambda}{2\pi} \int_{\psi(\beta)}^{4} \log_2 \left(1 + \frac{x}{L_{\alpha} \lambda^\frac{1}{2}} \right) \cdot \sqrt{\frac{4 - x}{x}} \, dx \quad \text{(42)}$$

where the second line substituted (40) and the definition:

$$L_{\alpha} \triangleq \lim_{M \to \infty} \frac{M}{\Lambda_{\alpha} \cdot \sqrt{\lambda}}. \quad \text{(43)}$$

Using (34) we have $L_{\alpha} = \frac{\Gamma(1 + \frac{1}{2}) \cdot \beta}{(\pi \Gamma(1 - \frac{1}{2}))^\frac{1}{2}}$. One can readily verify that the integral in the last line of (42) exists and is bounded for any $\alpha > 2$. Hence, the scaling law of the upper bound for spatial multiplexing WANET is linear with the number of antennas, $M$, for any value of $\lambda$ and $\beta$. Noticing that the lower bound, (11), has the same structure as the analyzed upper bound, we can deduce that the lower bound’s growth with the number of antennas is also linear. Hence, both the upper and lower bounds on the ERD scale linearly with the asymptotic growth in the number of antennas\footnote{In most interesting scenarios, the bound of the normalized ERD $(R(\lambda)/M)$ exists and hence also scales linearly with $M$. Yet, the proof and conditions for which the normalized ERD converges are out of the scope of this work.}.

The linear scaling of the bounds on the ERD with the number of antennas is similar to the known scaling of the bounds on the TC in MIMO WANETs \cite{8, 33}. However, the linear scaling for the TC metric was derived for WANETs that employed interference cancellation of neighboring transmissions, while our result only requires spatial multiplexing. This difference is important, because cancellation of neighboring transmission requires channel measurements of many neighbor transmissions, which is not needed for spatial multiplexing.

To conclude the MIMO WANET example, we summarize as follows: when the number of spatial streams decreases, the optimal density of active users increases. For a small number of antennas the distribution of the largest singular value with the number of antennas is also linear. Hence, both the upper and lower bounds on the ERD scale linearly with the asymptotic growth in the number of antennas.\footnote{In most interesting scenarios, the bound of the normalized ERD $(R(\lambda)/M)$ exists and hence also scales linearly with $M$. Yet, the proof and conditions for which the normalized ERD converges are out of the scope of this work.}
path loss factor, a two-dimensional optimization of both the density of active users and the number of streams is essential to achieve the maximum ERD. Nevertheless, the gain from the optimal spatial-multiplexing scheme compared to eigen-beamforming in density-optimized WANETs is much smaller than the potential capacity gain of single user MIMO. For large number of antennas, $M$, the bounds on the ERD scale as $M^2$ for eigen-beamforming, and as $M$ when the number of spatial streams is optimized.

V. CONCLUSION

We derived a novel upper bound on the ergodic rate density of random WANETs. The upper bound is very general and can support various transmission/reception schemes and general fading distributions.

The formula of the upper bound is shown to be similar to the formula of a recently published lower bound. This similarity is utilized for the quantification of the maximum gap between the bounds. The maximum gap was presented as a closed form expression which bounds the difference between the two bounds and is only a function of the path loss factor. Due to stability consideration, practical WANETs typically operate below the optimal network density. The gap expression shows that for such WANETs, the difference between the bounds is at most 50% for a path loss exponent in the range $2 < \alpha \leq 4$ (for example, for a path loss factor of 3 the gap between the bounds is shown to be at most 31%).

The usefulness and the simplicity of the bound was demonstrated by two applications: beamforming and spatial multiplexing. For the beamforming application we presented analytical expressions for the optimal density of active users and the optimal ERD as a function of the number of antennas. For the application of spatial-multiplexing we presented an upper bound that depends on the number of antennas, the number of spatial streams and the density of active users. We also proved that for large number of antennas, $M$, the bounds on the ERD scale as $M^2$ for eigen-beamforming, and as $M$ when the number of spatial streams is optimized.

APPENDIX A

PROOF OF THEOREM 1

Substituting (6) into (8) results in:

$$R_{f_v,f_y,f_o}(\lambda) = \lambda \cdot E_Y \left[ E_I \left[ \log_2 \left( 1 + \frac{Y}{I} \right) \right] Y \right]$$

$$\leq \lambda \cdot E \left[ \log_2 \left( 1 + E \left[ \frac{1}{I} \right] \right) Y \right]$$

(44)

where the second line used Jensen’s inequality. Denote the Probability Density Function (PDF) of the interference, $I$, by $f_I(t)$. Motivated by the mathematical formulation in [12, proof of proposition 2], we use the relation between the characteristic function of $I$ and the expectation of its inverse:

$$E \left[ \frac{1}{I} \right] = \int_0^\infty \left( \frac{1}{t} \right) f_I(t) dt$$

$$= \int_0^\infty \left( \int_0^\infty e^{-st} ds \right) f_I(t) dt$$

$$= \int_0^\infty \left( \int_0^\infty f_I(t) e^{-st} dt \right) ds$$

$$= \int_0^\infty E \left[ e^{-st} \right] ds$$

$$= \int_0^\infty \Phi(s) ds$$

(45)

where $\Phi(s) = E \left[ e^{-st} \right]$ is the characteristic function of $I$. The PDF of the interference, $f_I(t)$, has no known closed-form expression. However, its characteristic function is known and given by (e.g., [34], [35]):

$$\Phi(s) = \exp \left( -\lambda \int_0^\infty E \left[ 1 - e^{-sV^{\rho_s}} \right] 2\pi r dr \right)$$

$$= \exp \left( -\pi \lambda s \frac{\rho_s^2}{\pi} \right) \Gamma \left( 1 - \frac{2}{\alpha} \right).$$

(46)

Substituting (46) into (45) results in:

$$E \left[ \frac{1}{I} \right] = \int_0^\infty e^{-\pi \lambda E \left[ \frac{V^{\rho_s}}{2} \right]} E \left[ \frac{\rho_s^2}{2} \right] \Gamma \left( 1 - \frac{2}{\alpha} \right) ds$$

$$\geq \frac{\lambda}{2} \int_0^\infty t^{-\frac{2}{\alpha} - 1} e^{-t dt}$$

$$= \left( \frac{\pi \lambda E \left[ \frac{V^{\rho_s}}{2} \right]}{\rho_s^2} \right) \Gamma \left( 1 - \frac{2}{\alpha} \right)$$

$$= \frac{1}{\Lambda_\alpha \lambda^2}$$

(47)

where the second line substituted $t = \pi \lambda E \left[ \frac{V^{\rho_s}}{2} \right] E \left[ \frac{\rho_s^2}{2} \right] \Gamma \left( 1 - \frac{2}{\alpha} \right) x^{\frac{2}{\alpha}}$, the third line used the Gamma function definition, $\Gamma(z) \triangleq \int_0^\infty y^{z-1} e^{-y} dy$, and the Gamma function property, $\Gamma(z+1) = \Gamma(z) \cdot z$, and the last line substituted $\Lambda_\alpha$, defined in (15). Substituting (47) into (44) concludes the proof.

APPENDIX B

PROOF OF THEOREM 2

The left-hand side of the inequality in (24) is trivial from the definition of the lower bound. The theorem considers the density range of:

$$\lambda \leq \lambda^L = \lambda^UB \cdot \Omega_\alpha$$

(48)

where (21) was used. For $\lambda \leq \lambda^UB$ the upper bound, $R_{UB,f_v,f_y,f_o}(\lambda)$, increases with $\lambda$. Thus, using $\Omega_\alpha \leq 1$ (which is proved by Lemma 1 below) leads to:

$$R_{UB,f_v,f_y,f_o} \left( \lambda \cdot \Omega_\alpha \right)^{-1} \geq R_{UB,f_v,f_y,f_o}(\lambda) \forall \alpha \leq \lambda^LB \cdot \Omega_\alpha$$

(49)

Substituting (49) into (17) proves the right-hand side of the inequality in (24) and concludes the proof of the theorem.
Lemma 1: For any $\alpha > 2$, the value of $\Omega_{\alpha}$, defined in (18), satisfies: $\Omega_{\alpha} \leq 1$.

Proof of Lemma 1: Starting from the inequality [36]:
\[
\left(\frac{\Gamma(x + 1)}{\Gamma(ax + 1)}\right)^{\alpha} \leq 1, \quad \forall x \in [0, 1], \quad a \geq 1
\]  
and substituting $x = 1 - \frac{2}{\alpha}$ and $a = \frac{\alpha}{2}$ leads to:
\[
\left(\frac{\Gamma\left(2 - \frac{2}{\alpha}\right)}{\Gamma\left(\frac{\alpha}{2}\right)}\right)^{\frac{\alpha}{2}} \leq 1.
\]  
Raising both sides of (51) to the power of $2/\alpha$ and using Definition (18) concludes the proof of the lemma.

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Chapter 5

The Ergodic Rate Density of Slotted and Unslotted CSMA Ad-hoc Networks

The ergodic rate density of slotted and unslotted CSMA ad-hoc networks

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Abstract

The performance of random Wireless Ad-hoc Networks (WANETs) is primarily limited by their self-interference. The utilization of a decentralized Carrier Sensing Multiple Access (CSMA) protocol protects the participating receivers from the presence of strong interferers and enhances the performance compared to the simpler ALOHA protocol. In this work we analyze the Ergodic Rate Density (ERD) of slotted and unslotted CSMA WANETs in the small back-off probability regime. Our main result is the derivation of simple expressions which describe the ERD of CSMA WANETs as a function of the back-off probability, the path-loss exponent and the ERD of the same WANET when applying the ALOHA protocol. The ERD expressions for both the slotted and the unslotted variants are shown to grow with the back-off probability. For the slotted variant the gain of CSMA over ALOHA is equal to the back-off probability. On the other hand, for the unslotted variant this gain is smaller by a constant factor, which is within the range of 0.57 to 0.67 for all cases of practical interest. Simulation results validate the precision of the derived expressions and demonstrate their capability to predict the optimal system parameters with very good accuracy.

I. INTRODUCTION

Wireless Ad-hoc Networks (WANETs) are wireless networks that do not depend on a pre-existing infrastructure and therefore are suitable for many applications. WANETs are primarily characterized by multi-hop communication, offering scalability and flexibility.

The Medium Access Control (MAC) protocol is the set of rules that defines the access procedure to the shared medium. The MAC protocols of WANETs are decentralized by nature and have a significant impact on network performance. Traditionally, MAC protocols are divided into slotted type and unslotted type MAC protocols. Slotted MAC protocols are typically more efficient than their unslotted variants [1] but they require at least global or local time synchronization. Time synchronization in WANETs is not always guaranteed and may cost in additional hardware,
implementation complexity and protocol overhead. This leads to an engineering tradeoff between
the utilization of slotted and unslotted MAC protocols for WANETs. This tradeoff explains why
some of the standards support both variants (e.g., [2]). Note that the use of the term slotted to
c charact erize a MAC protocol has several meanings in the literature. In the following paper the
term slotted protocol refers to access protocols in which the time axis is divided into slots and
each transmission occupies a single slot.

The simplest decentralized MAC protocols are the unslotted ALOHA [3] and its slotted version
[1]. The pure randomness of the ALOHA protocols was shown to be inefficient for large amount
of terminals and to result in a throughput-stability tradeoff [4]. The Carrier Sensing Multiple
Access (CSMA) protocol [5] was shown to be more efficient and stable than the ALOHA protocol
and hence had been adopted by several leading communication standards (e.g., [2], [6]) as the
main access protocol. The usefulness of the CSMA protocol lies in its capability to ensure that
the mutual interference between terminals is limited to a power threshold. In situations where
the mutual interference has the potential to exceed the power threshold, access to the media of
the sensing terminal is denied and it backs off. In order to maximize the throughput of CSMA
WANETs the power threshold must be optimized as function of user density [7]–[10].

CSMA protocols are often compared to the simple ALOHA protocol. Blaszczyszyn et al.
used stochastic geometry based simulations to show the potential performance gain of unslotted
CSMA WANETs over slotted and unslotted ALOHA WANETs [11]. Their simulations showed
that the unslotted CSMA protocol always outperforms the ALOHA protocol. Several papers also
considered the impact of the CSMA protocol overhead on the network performance. The optimal
carrier sense threshold that maximizes the network throughput was shown to be larger compared
to the case in which the MAC overhead is not considered [12]. Another approach characterized
the network overhead by the resulting back-off probability, and showed that the performance
of slotted CSMA WANET grows exponentially with the back-off probability under an outage
model [13].

The main barrier for the analysis of large scale CSMA networks is the modeling of the random
position of active nodes. Repulsive point processes such as Matérn hardcore process [14], [15],
random sequential adsorption [16], and others [17]–[19] were considered as a position model.
A different approach [13], [20], [21], models the position of the interferers as a homogenous
Poisson Point Process (PPP) outside of a guard zone around a probe receiver.
The performance of random WANETs is typically evaluated by its spectral efficiency per unit area. Most results have been obtained using an outage model for each user rate (see [22] and references therein). Simpler expressions were obtained recently by the utilization of an ergodic rate model [23]. The Ergodic Rate Density (ERD) of ALOHA WANETs was lower bounded in [24] using simple and tight bounds. In particular, [24] showed that different system parameters have a similar impact on the ERD (see also [25], [26]). This interesting property of this lower bound prompted us to adapt the bounding techniques in [24] and to employ it to CSMA protocols as well.

In the following work we analyze the ERD of slotted and unslotted CSMA WANETs. The framework focuses on the access protocol and the physical layer processing and adopts a probabilistic simplified model to characterize the operation of higher communication layers. For the slotted CSMA variant the transmissions are restricted to the boundaries of predefined time slots whereas the unslotted variant is based on asynchronous access to the medium. The performance of both variants is evaluated as a function of the back-off probability. These expressions enable a fair comparison between the two variants and an inherent comparison of the CSMA protocols to the simple ALOHA scheme. Furthermore, the analysis allows for the evaluation of the system parameters that maximize the network performance.

The rest of this paper is organized as follows: Section II describes the system model. Section III introduces our main results and Section IV presents the proofs of the main results. Section V covers numerical and simulation results and concluding remarks are found in section VI.

II. System Model

A. Physical layer

We consider a decentralized wireless ad-hoc network (WANET), which utilizes a CSMA Medium Access Control (MAC) protocol. The locations of the transmitters in the network are random, and modeled by a two dimensional homogenous Poisson Point Process (PPP) with a density of $\lambda_p$. Following the CSMA protocol, described in subsection II-B, some of the nodes gain access to the medium and become active transmitters. Each active transmitter has a specific destination node, and together they form a pair. We mark the paired nodes by a common index; i.e., receiver $i$ is always paired with transmitter $i$. 
In the following we make several assumptions that facilitate the network analysis. In order to make this analysis more tractable, we first present a list of all major assumptions. We assume that:

1) All transmitters have data to transfer.
2) The distance between each transmitter and its paired receiver is equal to $d$ for all pairs, while the relative angle between a transmitter and its paired receiver is distributed uniformly over $[0, 2\pi]$. 
3) The effect of signaling miss-detections and collisions as well as the contribution of the signaling messages to the aggregate interference is negligible.
4) The channels and the location of nodes do not change during packet transmission.
5) The realizations of the PPP that models the node locations (before the CSMA MAC) at different transmission times is statistically independent, and so are the associated channel gains.

Assumption 2 is quite common for this type of analysis, and is justified by the operation of a reasonable routing protocol [14]. Assumption 3 holds if the packet duration is significantly longer than the signaling messages. Assumption 4 is relevant for low mobility WANETs (i.e., with large coherence time). Assumption 5 assumes in practical terms that the rate of transmission attempts for each user, denoted by $\mu$, is small compared to $1/T$ where $T$ is the packet duration. Note that limiting the rate of transmission attempts, $\mu$, does not necessarily lead to a low spatial density of transmission attempts, $\lambda_p$, since the latter is the product of the $\mu$, $T$ and the total density of nodes. Thus, we actually assume that the network is dense enough so that the high density of nodes allows Assumption 5 to hold. The low rate of transmission attempts is required to ensure that: a. the active interferers for each transmission attempt are likely to be completely different, and b. the time duration between transmission attempts is typically larger than the channel coherence time.

For the position of nodes in the network we use the Poisson bipolar model, [27]. In this model the transmitters’ locations, $\{x_i\} \subset \mathbb{R}^2$, are modeled as PPP and the receivers, $\{y_i\} \subset \mathbb{R}^2$, are positioned at a fixed distance $d$ and a uniformly randomly chosen orientation from their transmitters. We assume that each transmitter delivers its intended data packet over many transmissions (these transmissions can be divided in time or in the frequency domain).

Using assumption 5, the achievable rate at each transmission is affected by at most one
transmission of each interferer and is statistically independent of all other transmissions. Thus, for the performance analysis we can conveniently replace the Poisson bipolar model with a bipolar 3-dimensional Poisson rain model [28], [29] over space and time. In this model, an attempting pair is “born” at the time of its attempting event and “disappears” immediately after an unsuccessful access attempt or at the end of its transmission. To maintain a good approximation of the original Poisson bipolar model, we use the same space time density; i.e., the density of the rain bipolar model is set to \( \lambda = \mu \lambda_p \), where \( \mu \) is the rate of transmission attempts (in time) for each user. Denoting by \( \tau_i \) the attempting time of the \( i \)-th pair, the 3-D coordinates of the Poisson rain model of the \( i \)-th pair are given by \((x_i, y_i, \tau_i)\).

The power received at receiver \( i \) from transmitter \( j \) is given by:

\[
W_{i,j} = \rho D_{i,j}^{-\alpha} V_{i,j} \tag{1}
\]

where \( \rho \) denotes the transmission power, \( \alpha > 2 \) is the path-loss exponent, and \( V_{i,j} \) and \( D_{i,j} \) denote the channel fading and the distance between the \( j \)-th transmitter and the \( i \)-th receiver, respectively (i.e., \( D_{i,j} \triangleq ||x_i - y_j|| \) and \( D_{i,i} = d \) for all \( i \)). Using Assumption 5 the channel fading variables \( V_{i,j} \) are independent and identically distributed (i.i.d), and also statistically independent of all distance variables, \( D_{i,j} \) for all \( i, j \). In the following we will also use the notation \( V \) when we discuss the statistical nature of the Random Variable (RV) \( V_{i,j} \), bearing in mind that this is a single representative of a family of i.i.d RVs.

The operation of each transmitter is governed by two indicators. The access indicator, denoted by \( G_j \in \{0, 1\} \), indicates that the \( j \)-th pair has succeeded in gaining access to the media. The time indicator, denoted by \( Z_j(t) \in \{0, 1\} \), indicates that the \( j \)-th transmitter intends to transmit at time \( t \) (and in the case of a successful access it will transmit at that time). Thus, the multiplication of the two indicators is equal to 1 when the \( j \)-th transmitter is actually transmitting and 0 otherwise.

Using Assumption 2, the power of the desired signal measured by the \( i \)-th receiver, is equal to:

\[
S_i(t) = \rho d^{-\alpha} V_{i,i} Z_i(t). \tag{2}
\]

Note that the multiplication by \( Z_i(t) \) significantly reduces the number of non-zero elements in the sum (as \( Z_i(t) = 1 \) solely for one period of duration \( T \)). In other words, for each time \( t \) the summation can be taken only over \( \{j : j \neq i, Z_j(t) = 1\} \).
The aggregate interference measured by the \( i \)-th receiver at time \( t \) is given by:

\[
I_i(t) = \sum_{j \neq i} W_{i,j} Z_j(t) G_j = \sum_{j \neq i} \rho D_{i,j}^{-\alpha} V_{i,j} Z_j(t) G_j.
\] (3)

Denoting the contribution of the thermal-noise by \( \sigma^2 \), the Signal-to-Interference-and-Noise-Ratio (SINR) measured by the \( i \)-th receiver at time \( t \) is given by \( \frac{S_i(t)}{\sigma^2 + I_i(t)} \). Without loss of generality we set the desired transmission distance to be \( d = 1 \).

The Ergodic Rate Density (ERD) is defined as the density of the mutual information between transmitters and their paired receivers, i.e., the density of transmission attempts (per unit area and unit time) multiplied by the average mutual information of a transmission attempt. The ERD represents the maximal achievable network throughput, given that all transmissions use Gaussian signaling and all receivers treat interference as noise. The ERD can be approached by WANETs that utilize time diversity, frequency diversity or hybrid-HARQ (see [24] and references therein).

In mathematical terms, we denote by \( r_i \) the distance of receiver \( i \) from the origin and by \( \nu \) and \( \tau \) the analysis radius and period respectively. The ERD of the network is defined by:

\[
R = \lim_{\tau, \nu \to \infty} \frac{1}{\pi \nu^2 \tau} \sum_{i : r_i < \nu} \int_0^\tau \log_2 \left( 1 + \frac{G_i S_i(t)}{\sigma^2 + I_i(t)} \right) dt
\] (4)

where the number of symbols in a packet is assumed to be large and hence enables the use of an integral form instead of a sum over the discrete symbols.

B. CSMA MAC

Following Assumption 1, each transmitter waits for its attempt time, and then attempts to transmit the data. In the following, we will refer as an attempting transmitter to each transmitter starting from its attempt time and until the final result of its attempt. The transmitter, together with its intended receiver, are referred as an attempting pair and the attempting time of the \( j \)-th attempting pair is denoted by \( \tau_j \).

Utilizing a decentralized CSMA protocol, an access attempt starts with an interference test. The CSMA interference condition is guaranteed through the use of the RTS-CTS protocol: an attempting transmitter transmits a request to send (RTS) message and its paired receiver answers

\(^1\)Adapting the presented results to the case that \( d \neq 1 \) can be easily achieved by scaling the network (e.g., the received powers distribution in a network with parameters \( \lambda_p, \rho \) and \( d \) is identical to the power distribution in a network with \( \lambda'_p = \lambda_p \cdot d^2, \rho' = \rho d^{-\alpha} \) and \( d' = 1 \)).
with a clear to send (CTS) message. Each node constantly monitors the control messages of all other nodes. Thus, using channel reciprocity, each node has sufficient information to decide whether the transmission of the packet can be established without interfering with any prior-activated pair. The duration of the RTS and CTS messages is assumed to be long enough to ensure proper detection but very small compared to the packet duration (as indicated by Assumption 3).

The CSMA test is based on an average interference metric evaluated over the packet duration. In order to generalize the analysis we assume that the packet time for a transmission attempt at time \( \tau \) will start at \( g(\tau) \geq \tau \) and will end at \( g(\tau) + T \), where \( T \) is the packet duration, which is assumed to be fixed for all users. The function \( g(\cdot) \) will be different for the slotted and unslotted CSMA variants, and will be further discussed below. Thus, the relation between the user transmission indicator, \( Z_j(t) \), and its attempt time, \( \tau_j \), is given by:

\[
Z_j(t) = \begin{cases} 
1 & t \in [g(\tau_j), g(\tau_j) + T] \\
0 & \text{o.w.} 
\end{cases}
\] (5)

The CSMA mechanism is designed to limit the mutual interference among the active pairs so that all pairs at any time, \( t \), will satisfy:

\[
W_{jk} \cdot Q_{jk} \cdot G_j \cdot G_k < \rho \delta \cap W_{kj} \cdot Q_{kj} \cdot G_k \cdot G_j < \rho \delta
\] (6)

where \( \delta \) denotes the allowed interference threshold, and \( Q_{jk} \) is an overlap factor defined by:

\[
Q_{jk} \triangleq \frac{1}{T} \int_{g(\tau_j)}^{g(\tau_j) + T} Z_j(t)Z_k(t)dt.
\] (7)

For the \( j \)-th pair, arriving at \( \tau_j \) we say that the access attempt is successful if (6) is satisfied for all \( k \) such that \( \tau_k < \tau_j \). Thus, when the \( j \)-th pair gains access to the media, we set \( G_j = 1 \) and the \( j \)-th pair is referred as an active pair from \( g(\tau_j) \) to \( g(\tau_j) + T \). However, if (6) is not satisfied during an attempting event we say that a contention has occurred, the \( j \)-th transmitter backs off and we set \( G_j = 0 \). In this case, the data that was planned for a transmission is held until the next access attempt.

As each transmission has the same duration, \( T \), the density of active pairs at any given time is given by:

\[
\lambda_c = (1 - P_B(\lambda_c, \delta)) \mu T \lambda_p
\] (8)
where $P_B(\lambda_c, \delta) \triangleq \Pr(G_i = 0)$ is the back off probability. Note that although $\lambda_c$ is actually determined by $\lambda_p$ and $\delta$, we prefer to address the reverse relation, and consider the back-off probability as a function of $\lambda_c$ and $\delta$. Also note that (8) defines $\lambda_c$ as a function of $\delta$, $\lambda_p$, and the system parameters, $\mu$ and $T$.

The two variants of the CSMA protocol considered here use different transmission scheduling and hence differ solely by the function $g(\cdot)$. These two variants are:

1) **Unslotted CSMA**: The pairs begin their transmissions immediately after turning active, i.e.,

   \[ g(\tau) = \tau. \]

2) **Slotted CSMA**: The time axis is partitioned into slots of fixed duration, $T$, and each transmission falls within the boundaries of a single slot, i.e.,

   \[ g(\tau) = T \cdot \left\lceil \frac{\tau}{T} \right\rceil \]

   where $\left\lceil x \right\rceil$ denotes the smallest integer which is larger or equal to $x$.  

   Naturally, the performance of the CSMA protocol is compared to the performance of the ALOHA protocol. We note that the model in (6) can be degenerated into the simple ALOHA protocol by taking the limit as $\delta$ goes to infinity (and hence the back-off probability goes to zero).

   We also note that the ERD of an ALOHA network is identical for the slotted and un-slotted variants (as a verification, one can check that (4) depends only on the instantaneous distribution of the interference, and not on the statistical dependence between the interference in different times). Much of this work will be based on a recently published useful bound on the ERD of ALOHA WANETs [24]:

   \[ R_{ALOHA}^{ALOHA}(\lambda_c) \geq R_{LB}^{ALOHA}(\lambda_c) \quad (9) \]

where $R_{ALOHA}^{ALOHA}(\lambda_c)$ is the ALOHA ERD for an active user density of $\lambda_c$,

   \[ R_{LB}^{ALOHA}(\lambda_c) = \lambda_c e^{\frac{2}{\alpha}} - 1 \cdot E \left[ \log_2 \left( 1 + \frac{\rho Y}{\sigma^2 + C_\alpha \cdot \lambda_c^{\frac{\alpha}{2}}} \right) \right] \quad (10) \]

and

   \[ C_\alpha \triangleq \frac{2}{\alpha(\alpha - 2)^{\frac{1}{2}}} \cdot \left( \pi E \left[ (\rho V)^{\frac{\alpha}{2}} \right] \right)^{\frac{\alpha}{2}}. \quad (11) \]

\[ A \text{ different approach for this version is described in [13] and involves partitioning the slots to access and data sub-slots. If the transmission time is much larger than the duration of the signaling messages, the resulting performance is identical. The slotted CSMA protocol can also be considered a hybrid of a CSMA and a spatial Time Division Multiple Access (TDMA) protocol (e.g., [30], [31]).} \]
Note on notation: In this work we present approximations that are tight in the low back-off regime. An accurate presentation of these expressions relies heavily on the little $o$ notation: For a function $f(x)$ we say that $f(x) \in o(x)$ if \( \lim_{x \to 0} \frac{f(x)}{x} = 0 \).

III. MAIN RESULTS

The network model presented here has no penalty on back-off events. This is reasonable, because the effect of the back-off probability will always become negligible if the packet time is sufficiently large. However, the back-off probability is a major concern in any practical network, and a high back-off probability can have a major impact on protocol overheads, delays and network performance. To address the concerns of practical networks, the performance analysis throughout this work was carried out under the constraint of a small back-off probability. A second parameter which determines the performance of an ALOHA WANET is the active users density. Hence the bound is expressed as function of $P_B$ and $\lambda_c$ parameters. Our main results present asymptotic lower bounds on the ERD in the low back-off probability regime:

**Theorem 1:** The ERD of a slotted CSMA WANET is lower bounded by:

$$R_{S-CSMA} \geq R_{S-CSMA}^{LB}(\lambda_c, P_B) = (1 + P_B + o(P_B)) \cdot R_{ALOHA}^{LB}(\lambda_c)$$

where $P_B$ denotes the back-off probability.

*Proof:* See Subsection IV-C.

**Theorem 2:** The ERD of an unslotted CSMA WANET is lower bounded by:

$$R_{U-CSMA} \geq R_{U-CSMA}^{LB}(\lambda_c, P_B) = (1 + K_\alpha \cdot P_B + o(P_B)) \cdot R_{ALOHA}^{LB}(\lambda_c)$$

where

$$K_\alpha \triangleq \frac{2 + \alpha}{2 + 2\alpha}. \quad (12)$$

*Proof:* See Subsection IV-D.

The Theorems show that the performance of a CSMA WANET increases monotonically with the back-off probability, and in particular will always exceed the performance of an ALOHA WANET with the same density of active users$^3$. These two bounds are meaningful when the

$^3$This is of course a direct outcome of our model which neglect the impact of the CSMA protocol overhead on the network performance.
back-off probability is low, where the little \( o \) terms are negligible. This can be emphasized by the limit formulations of Theorems 1 and 2:

\[
\lim_{P_B \to 0} \frac{R_{LB}^{CSMA}(\lambda_c, P_B) - R_{LB}^{ALOHA}(\lambda_c, P_B)}{P_B} \geq R_{LB}^{ALOHA}(\lambda_c, P_B)
\]

\[
\lim_{P_B \to 0} \frac{R_{LB}^{UCSMA}(\lambda_c, P_B) - R_{LB}^{ALOHA}(\lambda_c, P_B)}{P_B} \geq K_\alpha \cdot R_{LB}^{ALOHA}(\lambda_c, P_B).
\]

Yet it is interesting to note that the Theorems give a good approximations for the actual ERD up to quite high back-off probability. In Section V we present numerical results that demonstrate that these approximations are useful even for back-off probabilities as high as \( 1/3 \).

For the slotted CSMA variant, Theorem 1 shows that the ERD gain of CSMA over ALOHA is equal to the back-off probability \( (P_B) \). The unslotted CSMA variant has a smaller gain of only \( K_\alpha P_B \). The factor \( K_\alpha \) decreases monotonically with \( \alpha \) and characterizes the unslotted CSMA degradation factor compared to the slotted variant. For \( \alpha > 2 \) this degradation is between \( 1/2 \) and \( 2/3 \).

It is interesting to note that the traditional comparison between the slotted and unslotted WANETs (which did not take spatial reuse into account) showed a capacity loss of up to 50\% for both ALOHA and CSMA [5]. This is a completely different result from the one presented above. The analysis here differs from the traditional approach in two main ways: the possibility of maintaining many simultaneous transmissions and the evaluation of ergodic rates. The difference in the results can be best observed by noting that for a small back-off probability the ERD of the two variants is nearly equal. Our result is more related to the result in [11], which showed that the same expression for \( K_\alpha \) also characterize the ratio between the performance of slotted and unslotted ALOHA under an outage rate model. The similarity results from the fact that the same average interference condition which is used here to declare back-off was used in [11] to declare outage. Furthermore, as discussed earlier, in the ALOHA network \( (P_B = 0) \) the ERD of the slotted and unslotted variants are identical.

Theorems 1 and 2 allow the evaluation of the ERD of a CSMA network from the ERD of an ALOHA network and the back-off probability. However, the evaluation of the back-off probability is also far from trivial. To facilitate the use of Theorems 1 and 2, the following Lemma presents the first order evaluation of the back-off probability.
Lemma 1: The back-off probability in a slotted CSMA WANET is given by:

\[ P_B = P^S_B(\lambda_c, \delta) = \pi \lambda_c E\left[V^{\frac{2\alpha}{\pi}}\right] \delta^{-\frac{\alpha}{\pi}} + o\left(\delta^{-\frac{\alpha}{\pi}}\right) \]  \hspace{1cm} (13)

and the back-off probability in an unslotted CSMA WANET is given by:

\[ P_B = P^U_B(\lambda_c, \delta) = \left(\frac{2\alpha}{\alpha + 2}\right) \pi \lambda_c E\left[V^{\frac{2\alpha}{\pi}}\right] \delta^{-\frac{\alpha}{\pi}} + o\left(\delta^{-\frac{\alpha}{\pi}}\right). \]  \hspace{1cm} (14)

Proof: See Subsection IV-A.

From a practical point of view, it is important to note that Theorems 1 and 2 together with Lemma 1 also allow the optimization of the network operating point. The expressions in Theorems 1 and 2 depend on the active users density, \( \lambda_c \), only through the ERD of the ALOHA network. Hence, the optimum active users density of the slotted and unslotted CSMA WANETs is identical to the optimum density of an ALOHA WANET which can be derived from the optimization of (10) (several examples can be found in [24]). Furthermore, as both \( R_{LB}^{S-CSMA}(\lambda_c) \) and \( R_{LB}^{C-CSMA}(\lambda_c) \) are monotonic increasing with the back-off probability, the maximal ERD is obtained when the back-off probability is set to its maximal allowed value. This is easily done by solving (13) or (14) to get the required value of the CSMA power threshold, \( \delta \). In Section V we provide closed form expressions for the parameters’ values that maximize the lower bounds. We also study the performance of a network that uses these optimized parameters (termed a bound optimized network) and show that it achieves performance that is close to the optimal.

Analysis Approach

The complete proof of the results presented above is given in the next section and in the appendices. As a motivation, we describe here the main steps of the analytical approach.

The analysis of CSMA ad-hoc networks presents some technical difficulties. In particular, the distribution of the active pairs on the plane is considered intractable\(^4\). Aiming to present an accurate analysis, we begin our mathematical derivations with the exact system model. Then, focusing on the low back-off probability regime, we neglect higher order terms of the derived expressions, using the little \( o \) notation.

\(^4\)The intractability of the active transmitters distribution can be demonstrated by inspecting the simplest case as \( d \to 0 \) and with no fading. In this case, the active transmitters distribution degenerates into the Matérn type III distribution [32]. However, although this distribution has been studied extensively in many contexts (e.g., [33] and references therein), the results are insufficient for system analysis.
The proof of Theorems 1 and 2 is based on the utilization of Jensen’s inequality over a conditional expectation of the ERD. To prove the Theorems we first find in Subsection IV-A the relation between the back-off probability and the system parameters (introduced above in Lemma 1). We next present in subsection IV-B a general lower bound on the ERD of CSMA WANETs (which holds for both the slotted and unslotted variants). The bound is proved using the law of total expectation, conditioning on the existence of a strong interferer, and lower bounding this conditional expectation using Jensen’s inequality. We finalize the proofs of Theorems 1 and 2 in Subsection IV-C and IV-D accordingly, where the ERD of slotted and unslotted CSMA WANETs is expressed as a function of the back-off probability, the path loss factor and the ERD of an ALOHA WANET with the same density of active users.

IV. PROOFS OF MAIN RESULTS

To evaluate the performance we use the homogeneity of the network. Using the ideas of [34], [35], one can easily verify that the CSMA WANET will converge to a homogenous steady state. Thus, we analyze the performance of the network taking user 0 as a probe receiver [36]. The probe receiver can be shown to be a typical point of the network [37]. Without loss of generality, we assume that this receiver is located at the origin and it satisfies $g(\tau) = 0$.

A. Proof of Lemma 1

1) The back-off probability in unslotted CSMA: To evaluate the back-off probability of the probe transmission attempt, we start by characterizing the number of transmitters that can cause a back-off, regardless of their success.

Denote by $N(\delta)$ the number of nodes that can cause a back-off when the probe receiver uses a power threshold of $\delta$. Taking into account that only users that are active at time $t = 0$ can cause back off to the probe receiver, the mean value of $N(\delta)$ for the $j$-th pair is given by:

$$\overline{N}(\delta) \triangleq E \left[ \sum_{j=1}^{\infty} Z_j(0) \cdot \Pr (W_{j,0} \cdot Q_{j,0} > \rho \delta \cup W_{0,j} \cdot Q_{0,j} > \rho \delta) \right]$$

$$= n_1 + n_2 - n_3$$

(15)

$$n_1 = E \left[ \sum_{j=1}^{\infty} Z_j(0) \cdot \Pr (W_{j,0} \cdot Q_{j,0} > \rho \delta) \right],$$

(16)
\[ n_2 = E \left[ \sum_{j=1}^{\infty} Z_j(0) \cdot \Pr(W_{0,j} \cdot Q_{0,j} > \rho\delta) \right] \]  

and

\[ n_3 = E \left[ \sum_{j=1}^{\infty} Z_j(0) \cdot \Pr\left((W_{j,0} \cdot Q_{j,0} > \rho\delta) \cap (W_{0,j} \cdot Q_{0,j} > \rho\delta)\right) \right]. \]  

Due to symmetry considerations, we have \( n_1 = n_2 \) and we can evaluate:

\[ n_1 = E \left[ \sum_{j=1}^{\infty} Z_j(0) \cdot \Pr\left(\rho X_{0,j} - \alpha \cdot V_{0,j} \cdot Q_{0,j} > \rho\delta\right) \right] \]

\[ = E \left[ \sum_{j=1}^{\infty} Z_j(0) \cdot \Pr\left(X_{0,j} < \left(\frac{V_{0,j} \cdot Q_{0,j}}{\delta} \right)^{\frac{1}{\alpha}}\right) \right] \]

where the first equality used (1). The multiplication of \( Z_j(0) \) in the left hand side of (19) can be interpreted as a thinning process. This thinning considers only pairs that began to be active at the time interval \([-T, 0]\). In this case, the interfering transmitters positions can be modeled by a two dimensional PPP with a spatial density of \( \mu T \lambda_p \). Hence, we can use the mean number of points inside a circle of radius \( VQ_j/\delta \) for a PPP and have:

\[ n_1 = \pi \mu T \lambda_p E \left[ Q_j^{\alpha} \left| Z_j(0) = 1 \right. \right] \]

\[ = \pi \mu T \lambda_p E \left[ \frac{V}{\delta} \right] \delta^{-\frac{2}{\alpha}} \]

where \( Q_j \mid Z_j(0) = 1 \) is a random variable that represents time overlap between the probe pair (that starts its transmission at time 0) and a typical contender that started to transmit at the time interval \([-T, 0]\). On the other hand, in Appendix A we show that for large enough \( \delta \), \( n_3 \) is proportional to \( \delta^{-3/\alpha} \) and hence we can write:

\[ p_3 = o\left(\delta^{-\frac{2}{\alpha}}\right). \]

Which results in:

\[ \bar{N}(\delta) = 2\pi \mu T \lambda_p E \left[ Q_j^{\frac{2}{\alpha}} \left| Z_j(0) = 1 \right. \right] E \left[ V^{\frac{2}{\alpha}} \right] \delta^{-\frac{2}{\alpha}} + o\left(\delta^{-\frac{2}{\alpha}}\right). \]

Thus, \( N(\delta) \) has a Poisson distribution with a parameter of \( \bar{N}(\delta) \). To discuss the probability of a back-off event, we partition the possible values of \( N(\delta) \) into three cases. The first case is \( N(\delta) = 0 \) which occurs with probability \( e^{-N(\delta)} \). In this case there is no node in the relevant vicinity, and hence the transmitter will not back off. The second case is \( N(\delta) \geq 2 \) which occurs
with probability \( \sum_{k=2}^{\infty} N(\delta)^k e^{-N(\delta)} / k! \). This case is the most complicated and we will not analyze it here. Instead, we note that for small \( N(\delta) \), this term is dominated by \( N^2(\delta) \), and hence, the back-off probability due to this term is bounded by \( o(N(\delta)) \). The last case is \( N(\delta) = 1 \) which occurs with probability \( N(\delta)e^{-N(\delta)} \). In this case the probe transmitter will back off if the relevant node is indeed active. The activity of this neighboring node is not affected by the activity of the probe receiver because it was decided prior to this time. Thus, the success probability of the neighbor transmitter is given by \( 1 - P_{UB} \). By combining the three cases, we can write the formula for the back-off probability as:

\[
P_{UB}(\lambda_c, \delta) = (1 - P_{UB}(\lambda_c, \delta))N(\delta)e^{-N(\delta)} + o(N(\delta))
\]

where Equality (a) is based on the little \( o \) notations for higher order terms (i.e., \( N(\delta) + o(N(\delta)) \) is equivalent to \( N(\delta)e^{-N(\delta)} + o(N(\delta)) \)), Equality (b) uses (21) and Equality (c) uses (8). Note that the last step in (22) allows us to write the back-off probability as a function of \( \lambda_c \) and not of \( \lambda_p \).

To conclude the proof, we present the following proposition:

**Proposition 1:** For the typical pair in the unslotted CSMA variant and for \( 0 \leq t \leq T \):

\[
E[Q_j^{2\alpha} | Z_j(t) = 1] = \frac{\alpha}{\alpha + 2} \left( 2 - \left( \frac{t}{T} \right)^{\frac{\alpha}{2} + 1} - \left( 1 - \frac{t}{T} \right)^{\frac{\alpha}{2} + 1} \right).
\] (23)

**Proof of Proposition 1:** See Appendix B.

We can now substitute (23) into (22) with \( t = 0 \), which leads to (14) and concludes the proof of this part of the Lemma.

2) **The back-off probability in slotted CSMA:** In [13] it was shown that a good approximation of the back-off probability is given by \( P_B \approx \pi \lambda_c E[Q_j^{2\alpha}]^{-\frac{\alpha}{2}} \). Following the same derivation while tracking the higher order terms better results in (13). The deviation of (13) from the derivation of [13] is not detailed here since it is very minor, and follows the same ideas described above for unslotted CSMA. This result concludes the proof of the Lemma.
B. ERD Lower Bound

We next present a general bound on the performance of CSMA WANETs:

Lemma 2: In the small back-off probability regime, a lower bound on the ERD of CSMA WANETs can be written as:

\[ R(\lambda_c, P_B) \geq R_{LB}^{CSMA}(\lambda_c, P_B) \]  \hspace{1cm} (24)

where

\[ R_{LB}^{CSMA}(\lambda_c, P_B) = (1 + o(P_B)) R_{LB}^{ALOHA}(\lambda_c) \cdot \frac{1}{T} \int_0^T e^{T(1+o(P_B))\pi \lambda_c E[v^2]} E\left[\frac{Q_j^2}{\sigma^2} \bigg| G_j(t) = 1\right] \frac{1}{\sigma^2} dt. \]  \hspace{1cm} (25)

Proof of Lemma 2: Due to the homogeneity of the WANET in time and space, the resulting process is ergodic, and the averaging in (4) converges to its expectation. Thus, the ERD expression, (4), can be evaluated by multiplying the pair density by the expected rate of the probe pair with respect to the corresponding Palm distribution. As the underlying point process is a homogenous PPP, we can replace the Palm expectation with the regular expectation with respect to the distribution of the PPP (see [38]):

\[ R(\lambda_c) = \lambda \cdot \int_0^T E \left[ \log_2 \left( 1 + \frac{G_0 S}{\sigma^2 + I(t)} \right) \right] dt, \]  \hspace{1cm} (26)

where \( S \) is the signal power, defined in (2), and \( I(t) \) is the interference power, defined in (3).

If the probe pair backed off \((G_0 = 0)\), its communication rate is equal to zero. Thus, it is sufficient to consider the conditional expectation of (26) given \( G_0 = 1 \), and to multiply the expression by the probability that the pair is successful (e.g., \( 1 - P_B \)). As all expectations in this section will be conditioned on the success of the probe pair, we prefer to shorten the notation, and define \( E^{G}[:\cdot] \) to denote the conditional expectation \( E[\cdot|G_0 = 1] \). As explained above, all expectations in this section (except of (26)) can be taken as conditional expectations. Any expectation in this section that does not use the \( E^{G}[:\cdot] \) notation (e.g., (25)) is taken on quantities that do not depend on the activity of the probe pair (i.e., for these expectations \( E[\cdot] = E^{G}[\cdot] \)).

Substituting (3), (8) and \( \lambda = \mu \lambda_p \) into (26) and using the \( E^{G}[\cdot] \) notation leads to:

\[ R(\lambda_c) = \frac{\lambda_c}{T} \cdot \int_0^T E^{G} \left[ \log_2 \left( 1 + \frac{S}{\sigma^2 + \sum_j W_j Z_j(t) G_j} \right) \right] dt \]

\[ = \frac{\lambda_c}{T} \cdot E^{G} \left[ \int_0^T E \left[ \log_2 \left( 1 + \frac{S}{\sigma^2 + \sum_j W_j Z_j(t) G_j} \right) \right] dt \right] \]
where the second line results from the law of total expectation.

Defining an arbitrary power threshold, denoted by \( \delta_0 \), and defining
\[
\chi(t) \define \max_{j \neq 0} (W_j G_j Z_j(t)),
\]
the ERD can be written as the sum of two complementary terms:
\[
R(\lambda_c) = \frac{\lambda_c}{T} \cdot E_S \left[ \int_0^T \Pr (\chi(t) \leq \rho \delta_0 | G_0 = 1) \cdot E^\varphi \left[ \log_2 \left( 1 + \frac{S}{\sigma^2 + \sum_{j>0} W_j Z_j(t) G_j} \right) \chi(t) \leq \rho \delta_0, S \right] dt \right] + \frac{\lambda_c}{T} \cdot E_S \left[ \int_0^T \Pr (\chi(t) > \rho \delta_0 | G_0 = 1) \cdot E^\varphi \left[ \log_2 \left( 1 + \frac{S}{\sigma^2 + \sum_{j>0} W_j Z_j(t) G_j} \right) \chi(t) > \rho \delta_0, S \right] dt \right].
\]

The power of the accumulative interference in the second term is unbounded and hence can be instantaneously very large. Thus, we drop the contribution of the user rate in this case, and lower bound it by zero. This bounding results in the following lower bound on the sum-rate:
\[
R(\lambda_c) \geq \frac{\lambda_c}{T} \cdot E_S \left[ \int_0^T \Pr (\chi(t) \leq \rho \delta_0 | G_0 = 1) \cdot E^\varphi \left[ \log_2 \left( 1 + \frac{S}{\sigma^2 + \sum_{j>0} W_j Z_j(t) G_j} \right) \chi(t) \leq \rho \delta_0, S \right] dt \right] \geq \lambda_c E_S \left[ \frac{1}{T} \int_0^T \Pr (\chi(t) \leq \rho \delta_0 | G_0 = 1) \cdot \log_2 \left( 1 + \frac{S}{\sigma^2 + E^\varphi \left[ \sum_{j>0} W_j Z_j(t) G_j \right] \chi(t) \leq \rho \delta_0} \right) dt \right] \geq \lambda_c E_S \left[ \frac{1}{T} \int_0^T \Pr (\chi(t) \leq \rho \delta_0 | G_0 = 1) \cdot \log_2 \left( 1 + \frac{S}{\sigma^2 + E^\varphi \left[ \sum_{j>0} W_j Z_j(t) G_j \right] \chi(t) \leq \rho \delta_0} \right) dt \right].
\]

where the second inequality uses Jensen’s inequality.

In this work we focus on the small back-off probability regime; i.e., when the threshold power \( \delta_0 \) is relatively small. Thus, keeping in mind that \( Q_j \leq 1 \ \forall j \), we limit the discussion to the case in which \( \delta > \delta_0 \). For this case, the expectation in the denominator is evaluated in Appendix C, resulting with:
\[
R(\lambda_c) \geq \lambda_c E_S \left[ \frac{1}{T} \int_0^T \Pr (\chi(t) \leq \rho \delta_0 | G_0 = 1) \cdot \log_2 \left( 1 + \frac{S}{\sigma^2 + (1-P_B+o(P_B)) \left( \frac{2\pi \rho}{\alpha^2} \right)^2} \right) dt \right].
\]
In order to bound the probability expression in the integral we introduce the following proposition:

**Proposition 2:** The probability that the maximal interference will satisfy the bound threshold is lower bounded by:

\[
\Pr (\chi(t) \leq \rho \delta_0 | G_0 = 1) \geq \exp \left( -\pi (1 - P_B + o(P_B)) \cdot \mu T \lambda_p E \left[ V\frac{2}{\pi} \right] \cdot \left( \delta_0^{\frac{2}{\alpha}} - \delta^{\frac{2}{\alpha}} E \left[ Q^\frac{2}{\alpha} \right] | Z_j(t) = 1 \right) \right).
\]

(30)

**Proof of Proposition 2:** See Appendix D.

Substituting (30) into (29) results in:

\[
R(\lambda_c, P_B) \geq \lambda_c E_S \left[ \frac{1}{T} \int_0^T \exp \left(-\pi (1 - P_B + o(P_B)) \cdot \mu T \lambda_p E \left[ V\frac{2}{\pi} \right] \cdot \left( \delta_0^{\frac{2}{\alpha}} - \delta^{\frac{2}{\alpha}} E \left[ Q^\frac{2}{\alpha} \right] | Z_j(t) = 1 \right) \right] \cdot \log_2 \left( 1 + \frac{S}{\sigma^2 + (1 - P_B + o(P_B)) \cdot \left( \frac{2\pi \rho}{\alpha - 2} E \left[ V\frac{2}{\pi} \right] \mu T \lambda_p \delta_0^{\frac{1}{\alpha} - \frac{2}{\alpha}} \right) } \right) dt \right]
\]

where the second equality substituted (8) and used:

\[
\left( 1 - P_B + o(P_B) \right) = 1 + o(P_B).
\]

(32)

We next present the following Proposition:

**Proposition 3:** For \( \xi \geq \frac{1}{2} \) and \( B, S, \sigma^2 \geq 0 \),

\[
\log \left( 1 + \frac{S}{\sigma^2 + \xi B} \right) \geq h(\xi) \cdot \log \left( 1 + \frac{S}{\sigma^2 + B} \right)
\]

(33)

where \( h(\xi) \) is defined as

\[
h(\xi) \triangleq \left( 1 - \left| \frac{1 - \xi}{\xi} \right| \right).
\]

(34)

**Proof of Proposition 3:** See Appendix E.
Substituting \( \xi = 1 + o(P_B) \), the condition \( \xi \geq \frac{1}{2} \) is trivially satisfied for the small back-off probability regime and \( h(\xi) = 1 + o(P_B) \). Combining proposition 3 and (31) leads to:
\[
R(\lambda_c, P_B) \geq \frac{(1 + o(P_B)) \lambda_c}{T} E_S \left[ \int_0^T \exp \left( -\pi (1 + o(P_B)) \right) \cdot \lambda_c E \left[ V^2 \right] \left( \delta_0^2 - \delta^2 E \left[ Q_j^2 \mid Z_j(t) = 1 \right] \right) \right] \cdot \log_2 \left( 1 + \frac{S}{\sigma^2 + \left( \frac{2\pi \rho}{(\alpha - 2) E \left[ V^2 \right] \lambda_c} \right) \int_0^T \right).
\]

(35)

Obviously, (35) is a lower bound which holds for any value of \( \delta_0 \). For the last step we follow [24] and choose:
\[
\delta_0 = \left( \frac{\alpha}{(\alpha - 2)} \pi E \left[ V^2 \right] \lambda_c \right)^{\frac{1}{2}}.
\]

(36)

Substituting (36) into (35) and using (10) concludes the proof of Lemma 2.

C. Proof of Theorem 1

For the slotted CSMA variant all transmissions are time-aligned and hence, \( Q_{0,j} \) can only take the values 0 or 1, i.e.,
\[
E \left[ Q_j^2 \mid Z(0) = 1 \right] = 1.
\]

(37)

Recognizing the similarity between the exponent in (25) and the back-off probability given in (13), and substituting (37) leads to:
\[
\frac{P_{LB}^S}{P_{ALOHA}^S} = (1 + o(P_B^S)) e^{(1+o(P_B^S)) P_B^S} \geq (1 + o(P_B^S))(1 + (1 + o(P_B^S)) P_B^S) = 1 + P_B^S + o(P_B^S)
\]

(38)

where the last equality simplifies the first order term.
D. Proof of Theorem 2

We start the proof by substituting (23) into (25), which leads to:

\[
\frac{R^U_{LB}}{R^ALOHA_{LB}} = \frac{1 + o(P^U_B)}{T} \int_0^T \exp \left( \frac{(1 + o(P^U_B)) \pi \lambda \alpha E[V^2] \delta^{-\frac{\alpha}{2}}}{\alpha + 2} \right) \cdot \left( 2 - \left( \frac{t}{T} \right)^{\frac{2}{\alpha} + 1} - \left( 1 - \frac{t}{T} \right)^{\frac{2}{\alpha} + 1} \right) \, dt
\]

\[
\leq (a) \frac{1 + o(P^U_B)}{T} \int_0^T \exp \left( \frac{(1 + o(P^U_B)) (P^U_B + o(P^U_B))}{2} \right) \cdot \left( 2 - \left( \frac{t}{T} \right)^{\frac{2}{\alpha} + 1} - \left( 1 - \frac{t}{T} \right)^{\frac{2}{\alpha} + 1} \right) \, dt
\]

\[
\leq (b) (1 + o(P^U_B)) \int_0^1 \exp \left( \frac{(P^U_B + o(P^U_B))}{2} \right) \cdot \left( 2 - x^{\frac{2}{\alpha} + 1} - (1 - x)^{\frac{2}{\alpha} + 1} \right) \, dx
\]

where Equality \((a)\) used (14) and Equality \((b)\) used the substitution \(t = T \cdot x\) and dropped the higher order term \(o^2(P^U_B)\).

Equation (39) can be further lower bounded using:

\[
\frac{R^U_{LB}}{R^ALOHA_{LB}} \geq (a) (1 + o(P^U_B)) \cdot e^{\frac{(P^U_B + o(P^U_B))}{2}} \int_0^1 \left( 2 - x^{\frac{2}{\alpha} + 1} - (1 - x)^{\frac{2}{\alpha} + 1} \right) \, dx
\]

\[
\equiv (b) (1 + o(P^U_B)) \cdot e^{(P^U_B + o(P^U_B)) \cdot K_\alpha}
\]

\[
\geq (c) (1 + o(P^U_B)) \cdot (1 + (P^U_B + o(P^U_B)) \cdot K_\alpha)
\]

\[
\equiv (d) 1 + K_\alpha \cdot P^U_B + o(P^U_B)
\]

where Inequality \((a)\) used Jensen’s inequality \(E[e^{h(x)}] \geq e^{E[h(x)]}\), Equality \((b)\) substituted \(K_\alpha\) (defined in (12)) as the solution of the integral, Inequality \((c)\) used \(e^x \geq 1 + x\) and Equality \((d)\) simplified the higher order terms. Note that for a small back-off probability, the inequalities \((a)\) and \((c)\) in (40) are tight.

V. Numerical Results

In the following section we demonstrate our main results and compare them to numerical simulations. All simulations were performed in the interference limited regime in which the
contribution of the thermal noise was neglected. The channel was characterized by Rayleigh fading, the path-loss exponent was $\alpha = 3$ and $\mu T = 1$. Note that for the Rayleigh distribution $E[V^2] = \Gamma(1 + \frac{2}{\alpha})$. All simulated results were obtained by performing Monte-Carlo simulations averaged over 5,000 network realizations. Each realization was simulated using the PHY model and the CSMA protocols that are described in Section II.

As explained in section III, a large back-off probability increases the protocol overhead and the average delay. Hence, practical CSMA WANETs place constraints on the maximum back-off probability, denoted here by $\eta$. In order to optimize network performance we define the set $S(\eta)$, which include all pairs $(\lambda_c, \delta)$ that satisfy the back-off probability constraint, i.e.,

$$S(\eta) \triangleq \{ (\lambda_c, \delta) : P_B(\lambda_c, \delta) \leq \eta \}. \quad (41)$$

The maximum ERD as function of the maximum allowed back-off probability is defined as:

$$R_{\text{max}}(\eta) \triangleq \max_{(\lambda_c, \delta) \in S(\eta)} R(\lambda_c, \delta). \quad (42)$$
Fig. 1 depicts the CSMA gain over ALOHA, i.e., the ratio of the maximum ERD in a CSMA network and the maximal ERD in an ALOHA network (maximized over $\lambda_p$ with $P_B = 0$), where both networks have the same density of active users. The ERD ratio is depicted as a function of the allowed back-off probability, for the slotted and unslotted variants. As a reference, the figure also shows the CSMA gain over ALOHA, predicted by Theorem 1 for slotted CSMA (i.e., $1 + P_B$) and by Theorem 2 for unslotted CSMA (i.e., $1 + K_\alpha P_B$), where setting $\alpha = 3$ into (12) leads to $K_\alpha = 0.625$. As can be seen, Theorems 1 and 2 give a good characterization of the ERD at low back-off probabilities. Furthermore, the predicted gain of the unslotted CSMA gives a very good approximation for the actual gain for a back-off probability as high as 0.35 while the slotted CSMA curves match even for a back-off probability as high as 0.6.

In order to further demonstrate the usefulness of Theorems 1 and 2, Fig. 2 depicts the simulated ERD as function of the active user density for five networks: ALOHA, slotted CSMA with back-
off probabilities of 0.15 and 0.3 and unslotted CSMA with back-off probabilities of 0.24 and 0.48. These cases were chosen since they correspond to the gain ratio between the two variants, i.e., $\frac{0.15}{0.24} = \frac{0.3}{0.48} = K_\alpha = 0.625$. As shown in the figure, the ERD curves of the slotted CSMA variant for $P_B = 0.15$ and the unslotted variant for $P_B = 0.24$ merge and have a gain of 15-17% over the ALOHA ERD. The ERD curves of the slotted CSMA variant for $P_B = 0.3$ and the unslotted variant for $P_B = 0.48$ differ slightly (up to 2% offset) since the analysis assumed a small back-off probability. However, the curves still show quite a good match to the predicted gain (a gain of 28-32% compared to the predicted gain of 30%).

As stated in Section III, the bounds also allow an accurate optimization of the network parameters. The optimal active user density of both CSMA variants is identical to the optimal
Fig. 4. The simulated maximum ERD of CSMA and bound optimized CSMA as a function of the back-off probability for the slotted and unslotted variants.

Density of an ALOHA WANET\textsuperscript{5} with the same active user density, i.e.,

\[ \lambda_c^* \triangleq \arg \max_{\lambda_c} R_{\text{LB}}^{\text{ALOHA}} (\lambda_c). \]  

Hence, using (8) the optimal participating user density for both CSMA variants given a back-off probability is:

\[ \lambda_p^* = \frac{\lambda_c^*}{\mu T (1 - P_B)}. \]  

From (13) and (14) respectively we can find the optimal power threshold for the slotted variant:

\[ \delta^* = \left( \frac{\pi \lambda_c^* E \left[ V^2 \right]}{P_B} \right)^{\frac{2}{\alpha}}. \]  

and for the unslotted variant:

\[ \delta^* = \left( \frac{2 \alpha \pi \lambda_c^* E \left[ V^2 \right]}{(\alpha + 2) \cdot P_B} \right)^{\frac{2}{\alpha}}. \]  

\textsuperscript{5}See for example [24] for a discussion of the optimal density in various ALOHA networks.
In the remainder of this section, we illustrate the usefulness of the presented results for network optimization. In order to obtain the optimal parameters of the network we first found $\lambda^*_c$ as defined in (43). We then substituted $\lambda^*_c$ and the planned back-off probability into (44) for $\lambda^*_p$ and into (45) and (46) for $\delta^*$ (for the slotted and unslotted variants respectively). The performance of a CSMA WANET that utilizes the optimal parameters $\lambda^*_p$ and $\delta^*$ is referred in the following as *bound optimized CSMA*.

Fig. 3 demonstrates the accuracy of the back-off probability formulas ((13) and (14)). The x-axis shows the planned back-off probability whereas the y-axis shows the actual back-off probability from Monte-Carlo simulations. The parameters for the evaluation of each point were obtained by selecting the parameters that maximized the ERD for a specific planned back-off probability, using equations (43), (44), (45) and (46). As can be seen from the figure, the back-off expressions are quite accurate and hold for large back-off probabilities up to 0.5.

To summarize the usefulness of the optimized parameters, Fig. 4 depicts the maximum ERD of the CSMA, (42), and the bound-optimized CSMA for the slotted and unslotted variants respectively. The bound optimized CSMA was plotted by using the optimal parameters, $\lambda^*_c$ and $\delta^*$ and evaluating the ERD, (4), by Monte-Carlo simulations. The maximal performance were evaluated by brute force optimization using Monte-Carlo simulations. The figure shows that the curves of the optimal CSMA and the bound-optimized CSMA are very close. This suggests that the optimal parameters, that were derived from the bounds, can be very useful for CSMA WANET optimization.

VI. Conclusion

We introduced novel lower bounds on the ERD of slotted and unslotted CSMA WANETs in the small back-off probability regime. For both variants, the gain of the CSMA was shown to be linear with the back-off probability. However, the bounds show that the unslotted variant requires a higher back-off probability to achieve the same ERD as the slotted variant. The required additional back-off probability is in the range of 50%-100% depending on the exact path-loss exponent. The expressions for the lower bounds’ are presented in terms of the ERD gain of a CSMA network over an ALOHA with the same density of active users.

The novel lower bounds were shown to be useful for the optimization of network parameters. The optimal active user density was shown to be identical to the optimal density of ALOHA
WANETs. The analysis also enabled a simple evaluation of the optimal density of transmission attempts (including attempts that will result in back-off) and the optimal sensing power threshold. Simulation results showed that the CSMA gain, predicted by the lower bounds, is very accurate even up to a back-off probability of 0.5, and that the derived expressions for the optimal parameters achieve close to optimum performance.

APPENDIX A

THE DOUBLE COLLISION PROBABILITY

In this appendix we analyze the case in which a pair performs a back-off due to a violation of both the CTS and the RTS conditions. In this scenario the $j$-th pair is trying to become active at time $\tau_j$ and backs-off due to the $k$-th pair where $\tau_j > \tau_k$. The $j$-th receiver suffers from a CTS condition violation denoted by the interference from the $k$-th transmitter while simultaneously the $j$-th transmitter performs an RTS condition violation over the $k$-th receiver. To be more specifically, our target in this Appendix is to evaluate the order of $\delta$ in the probability of this event.

Fig. 5 illustrates this event, where the squares indicate receivers and the triangles indicate transmitters. The probe receiver and transmitter are depicted at the center of circles which represent their CTS and RTS protection areas respectively (the different radii illustrate the effect of fading). The two other nodes form a pair that simultaneously violates the CTS and the RTS conditions of the probe pair and hence perform a back-off.

In order to calculate the probability of simultaneous CTS and RTS violations we multiply the CTS violation probability by the conditional probability of the RTS violation given a CTS violation. As shown in Fig. 5 the interfering transmitter is located inside of the RTS circle and it paired receiver is located at a distance $d$ from it. Thus, the conditional probability is equal to the angle of the illustrated sector divided by $2\pi$. This angle is clearly proportional to the diameter of the RTS protection circle which is equal to $\delta V^\frac{1}{2}$. Since the probability of a CTS violation is proportional to $\delta^{-\frac{1}{2}}$ (see (19)), the probability of a joint CTS and RTS violation event is proportional to $\delta^{-\frac{3}{2}}$. 
APPENDIX B
PROOF OF PROPOSITION 1

For the unslotted CSMA protocol the distribution of the overlap factor, \( Q_j \), varies during the reception of a packet. Given that \( Z_j(t) = 1 \), the distribution of the transmission start time of the \( j \)-th user’s is uniformly distributed over \((t-T,t]\). Recall that the probe pair starts its transmission at time 0; thus, the Probability Density Function (PDF) of the overlap factor of the \( j \)-th pair and the probe pair for \( 0 \leq t \leq T \), is given by:

\[
f_{Q_j|Z_j(t)=1}(q; t) = U\left(q - \frac{t}{T}\right) + U\left(q - \frac{T - t}{T}\right) - 2 \cdot U\left(q - 1\right)
\]

where \( U(x) \) is the step function (equal to 1 when \( x \geq 0 \) and 0 otherwise).
Using (47) the expectation in (23) results in:
\[
E\left[Q_j^2 \left| \alpha_j \right| = 1 \right] = \int_{-\infty}^{\infty} q^2 f_{Q_j|Z_j(t)=1}(q; t) dq \\
= \int_{t}^{1} q^2 dq + \int_{0}^{1} q^2 dq \\
= \frac{\alpha}{2 + \frac{1}{\alpha}} \left( 2 - \left( \frac{t}{T} \right)^{\frac{2}{\alpha}+1} - \left( 1 - \frac{t}{T} \right)^{\frac{2}{\alpha}+1} \right).
\]

\[\Box\]

**Appendix C**

**Aggregate Interference**

In this appendix, we evaluate the conditional expectation of the interference: \(E^G \left[ I(t) \left| \chi(t) \right. \right. \leq \rho \delta_0 \]. We analyze the average aggregate interference measured at a probe receiver:

\[
E^G \left[ I(t) \left| \chi(t) \right. \right. \leq \rho \delta_0 \] = E^G \left[ \sum_{i \neq 0} W_i Z_i(t) G_i \left| \chi(t) \right. \right. \leq \rho \delta_0 \right] \]

\[
= E^G \left[ \sum_{i \neq 0} \rho D_i^{-\alpha} V_i Z_i(t) G_i \left| \chi(t) \right. \right. \leq \rho \delta_0 \right] \\
= E^G \left[ \sum_{i \neq 0} \rho D_i^{-\alpha} V_i Z_i(t) \cdot \tilde{G}(D_i, V_i) \right] \tag{48}
\]

where \(\chi(t)\) is defined in (27), Equality \((a)\) uses (3), Equality \((b)\) uses (1) and Equality \((c)\) uses the law of total expectation (conditioning on \(D_i, V_i\)) and the definition \(\tilde{G}(D_i, V_i) \triangleq E^G \left[ G_i \left| D_i = x, V_i = v, \chi(t) \right. \right. \leq \rho \delta_0 \right] \) for user \(i\) that satisfies \(Z_i(t) = 1\). This expression evaluates the probability of an attempting pair to become active assuming that the probe pair is active.

We distinguish among three different types of attempting pairs. Pairs of the first type are the pairs with \(W_i = D_i^{-\alpha} V_i > \rho \delta_0\). Due to the conditioning by \(\chi(t) \leq \rho \delta_0\), we know that such users must satisfy \(G_i = 0\). In particular, this type includes all nodes that backed off directly due to the CTS condition of the probe receiver. Thus, \(\tilde{G}(D_i, V_i) = 0\) for any pair with \(D_i^{-\alpha} V_i > \rho \delta_0\). Pairs of the second type are the pairs that did not back off due to the CTS condition; however their transmitter is close to the probe receiver. The probability that these pairs will become active without the knowledge about the activity of the probe pair is \(1 - P_B + o(P_B)\). However, since we know that user 0 is active, and its transmitter may cause them to back off as a result
of its RTS condition, the probability that these pairs will become active is upper bounded by 
\[ 1 - P_B + o(P_B) \]. The third type covers all the other pairs which did not back off due to the probe 
pair, but may back off in response to any other pair. The probability that these pairs will back off 
depends on pair 0’s activity with higher orders of \( P_B \). Hence, in the small back-off probability 
regime, the probability of these pairs to become active can be considered as independent in the 
probe receiver activity and therefore is given by \( 1 - P_B + o(P_B) \).

The activity probability (due to the contribution of the second and third types of pairs) is 
hence upper bounded by 
\[ \bar{G}(\rho \delta, x, v) < 1 - P_B + o(P_B) \].

Thus, we have 
\[ E^G[I(t)|\chi(t) \leq \rho \delta_0] \leq (1 - P_B + o(P_B)) \cdot \psi \] where \( \psi \triangleq E \left[ \sum_j W_j Z_j(t) \mid \forall j, W_j Z_j(t) \leq \rho \delta_0 \right] \).

Noting that the density of nodes with \( Z_j(t) = 1 \) is equal to \( \mu T \lambda_p \) and using the known closed form expression, [24], for the aggregate interference 
from a PPP outside of a power guard zone lead to:

\[ E^G[I(t)|\chi(t) \leq \rho \delta_0] \leq (1 - P_B + o(P_B)) \cdot \left( \frac{2 \pi \rho \alpha}{2 - E[V_{j}^{2}] \mu T \lambda_p \delta_0^{1 - \frac{\alpha}{2}}} \right). \]

\section*{Appendix D}

\textbf{Proof of Proposition 2}

To evaluate the probability for a strong interferer, we first calculate the mean number of 
attempting transmitters that violate the power threshold condition, i.e., \( W_j Z_j(t) > \rho \delta_0 \). Recalling 
that any pair with \( W_i Q_i > \rho \delta \) will back-off due to the CTS of the probe receiver, the calculation 
at this stage does not consider the probability of the transmitters to turn active. The average 
number of attempting pairs that can exceed the threshold is given by:

\[ \overline{N}(\delta, \delta_0, t) \overset{(a)}{=} E \left[ \sum_{j=1}^{\infty} \Pr(W_j Z_j(t) > \rho \delta_0 \cap W_j Q_j < \rho \delta) \right] \]

\[ \overset{(b)}{=} E \left[ \sum_{j: Z_j(t) = 1} \Pr(\delta^{\frac{\alpha}{2}} V_j^{\frac{1}{\alpha}} Q_j^{\frac{1}{\alpha}} < D_j < \delta_0^{\frac{\alpha}{2}} V_j^{\frac{1}{\alpha}} \mid Z_j(t) = 1) \right] \]

\[ \overset{(c)}{=} \pi \mu T \lambda_p E \left[ V_{j}^{2} \right] \left( \delta_0^{\frac{\alpha}{2}} - \delta^{\frac{\alpha}{2}} \right) \left( Q_j^{\frac{\alpha}{2}} \mid Z_j(t) = 1 \right) \] (49)

where (a) uses (6), (b) uses (1) and exclusion of the non contributing terms from the sum, 
and (c) uses the mean number of points inside a ring for a PPP distribution (the density of
transmitters with $Z_j(t) = 1$ at any time $t$ is given by $\mu T \lambda_p$. Thus, the probability of at least one attempting transmitter to violate the condition at time $t$ is given by $1 - e^{-N(\delta, \delta_0, t)}$.

In the second stage, we want to take the transmitter activity ($G_j$) into account and derive a lower bound on the probability $\Pr(\chi(t) \leq \rho\delta_0 | G_0 = 1)$. If there is only one transmitter that exceeds the threshold, its activation probability is given by $E^G[G_j]$. If there are more than one transmitter, the threshold will be exceeded if at least one of the transmitters remains active. Thus, the probability $\Pr(\chi(t) \leq \rho\delta_0 | G_0 = 1)$ is equivalent to the probability that all the transmitters that exceeded the threshold of the probe receiver at time $t$ will not be active. For the case of independent activation of $k$ transmitters we can lower bound $\Pr(\chi(t) \leq \rho\delta_0 | G_0 = 1) \leq \left(1 - \max_j E^G[G_j]\right)^k$ where the inequality adopted a worst case approach and assumed that all transmitters were activated with a probability of $\max_j E^G[G_j]$. However, as discussed in Appendix C, transmitters that exceed the threshold are spatially correlated and hence their first order term of the back-off probability is higher than for the case of independent pairs. A lower bound on the probability of the probe pair to maintain a power threshold is:

$$\Pr(\chi(t) \leq \rho\delta_0 | G_0 = 1) \geq \sum_{k=0}^{\infty} \left(1 - \max_j E^G[G_j]\right)^k \cdot \Pr(N(\delta, \delta_0, t) = k)$$

where the equality uses the expansion series $e^{x} = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ and the PDF of a Poisson distribution with density of $\lambda$, i.e., $\Pr(x = k) = \frac{\lambda^k e^{-\lambda}}{k!}$.

As in the previous appendix, by dividing the pairs into three types we know that the probability of an attempting transmitter to turn active is upper bounded by $1 - P_B + o(P_B)$. Thus, the probability of these transmitters to satisfy the arbitrary threshold condition is lower bounded by:

$$\Pr(\chi(t) \leq \rho\delta_0 | G_0 = 1) \geq e^{-\left(1 - P_B + o(P_B)\right) \cdot N(\delta, \delta_0, t)}$$

Substituting (49) into (51) concludes the proof of this proposition.

APPENDIX E
PROOF OF PROPOSITION 3

We start the proof by introducing the following Lemma:
Lemma 3: For \( c > -1, \phi > 0 \) the following inequality holds:
\[
\frac{|\log (1 + \phi(1 + c)) - \log (1 + \phi)|}{\log(1 + \phi)} \leq |c|.
\]
(52)

Proof of Lemma 3: We distinguish between several ranges of \( c \). For the cases where \( c = -1 \) and \( c = 0 \) equation (52) is trivial. For the cases of \( c > 0 \) and \( -1 < c < 0 \) equation (52) results in:
\[
(1 + \phi(1 + c)) - (1 + \phi)^{1+c} \leq 0
\]
(53)
and
\[
(1 + \phi)^{1+c} - (1 + \phi(1 + c)) \leq 0
\]
(54)
respectively. We conclude the proof by noting that substituting \( a = 1 + c \) into equations (53) and (54) results in the general Bernoulli inequalities [39]:
\[
(1 + \phi)^a > 1 + a\phi, \forall \phi > 0, a > 1
\]
(55)
and
\[
(1 + \phi)^a < 1 + a\phi, \forall \phi > 0, 0 < a < 1
\]
(56)
respectively.

From Lemma 3 we can deduce that:
\[
\forall |c| < 1, \phi > 0: \log (1 + \phi(1 + c)) \geq (1 - |c|) \cdot \log(1 + \phi).
\]
(57)

Substituting \( \phi \triangleq \frac{S}{\sigma^2 + B} \) and \( c \triangleq \frac{1 - \xi}{\sigma^2 + B + \xi} \) into (57), while conditioning \( \xi \geq \frac{1}{2}, \sigma^2 \geq 0, B > 0, S > 0 \) results in:
\[
\log \left(1 + \frac{S}{\sigma^2 + \xi B}\right) \geq \left(1 - \frac{|1 - \xi|}{\frac{\sigma^2}{B} + \xi}\right) \cdot \log \left(1 + \frac{S}{\sigma^2 + B}\right)
\]
\[
\geq \left(1 - \frac{|1 - \xi|}{\xi}\right) \cdot \log \left(1 + \frac{S}{\sigma^2 + B}\right).
\]
(58)
Substituting (34) into (58) concludes the proof.
REFERENCES


Chapter 6
Discussion and Conclusions

This doctoral dissertation introduced a cross layer analysis of random wireless ad-hoc networks (WANETs). It investigated the impact of different Medium Access Control (MAC) protocols and smart antenna techniques on the performance of random WANETs. The dissertation is composed of four manuscripts covering different aspects of this subject.

All four manuscripts share a common framework, aiming to shed light on different aspects of the field of research. The main contribution is the introduction of closed form expressions which bound the Area Spectral Efficiency (ASE) of WANETs. These expressions were revealed to be very informative and enabled the derivation of many insights.

The dissertation focused on the MAC and PHY sub-layers of WANETs. Future research can expand these bounds to investigate the impact of upper layers (e.g., routing, application) on the performance of WANETs. The analytical tools, spatial models and bounding techniques can be adopted in the future for the optimization of the end-to-end delay, power control schemes and nodes cooperation.
Bibliography


