Abstract—We analyze the scaling laws of the Ergodic Rate Density (ERD) in ALOHA Wireless Ad-hoc Networks (WANETs) that utilize Multiple-In-Multiple-Out (MIMO) techniques. We present a novel lower bound that support spatial-multiplexing, interference-cancellation and beamforming. The bound is used to present expressions of the optimum system parameters as function of the number of antennas and the path-loss factor. When the density of active users can be tuned to maximize the performance, we show that the ERD scales linearly with the number of antennas and that single stream transmission optimizes the network performance. When the active users density is limited and it’s below 70% of the optimal density, spatial multiplexing was shown to have better performance than single stream transmission. Simulation results show the practicality of the novel bound and the accuracy of the estimated parameters.

I. INTRODUCTION

Wireless Ad-hoc Networks (WANETs) offer simplicity and flexibility that make them suitable for many practical applications. The main advantage of WANETs is their capability to be deployed without the need for infrastructure. These networks rely on decentralized channel access protocols such as ALOHA [1] and Carrier Sense Multiple Access (CSMA) [2], [3].

In recent years, the analysis of WANETs was the subject of many researches. In particular, random network analysis has succeeded to attain considerable insights on the performance of WANETs. This type of analysis enables performance evaluation of WANETs without considering specific user locations.

The most popular model for the position of users in random WANETs is the homogeneous Poisson Point Process (PPP) [4]. In this model the number of users in a finite area has a Poisson distribution, and their locations are uniformly distributed over the area. For the analysis of random WANETs, we follow the approach that uses exact modeling of the physical and Media Access Control (MAC) layers, together with a simplified abstraction of the higher communication layers.

The performance of random WANETs is typically measured by the mean communication-rates per unit area. The Transmission Capacity (TC) metric assumes a fixed communication rate among all pairs under an outage probability constraint. On the other hand the Ergodic Rate Density (ERD) metric considers the communication rate of each pair as the mutual information between the transmitted signal and the received signal given the interferers’ activity. The ERD was shown to be superior to the TC while requiring more advanced schemes, such as time diversity, frequency diversity, [5], or incremental-redundancy hybrid automatic repeat request (IR-HARQ), [6].

A recent work presented general and simple lower bounds on the ERD of random ALOHA WANETs [7] which can support various pre/post processing schemes. In particular, a bound that considered the utilization of an interference cancellation scheme was shown to be tight, and to converge to the ERD if the number of cancelled interferers is large enough. Although this bound was used to analyze several applications of Multiple-In-Multiple-Out (MIMO) communication, [7] did not consider the spatial-multiplexing scheme.

In this work we expand the lower bound of [7] on the ERD of random WANETs to include spatial-multiplexing in addition to the interference-cancelation and beamforming schemes. The novel lower bound enables the derivation of closed form expressions of the optimal network parameters.

Prior work evaluated the potential contribution of spatial-multiplexing to the performance of random WANETs under the TC metric (the TC outage constraint was enforced on each spatial stream). An interesting result [8] showed that when receivers can perform interference-cancellation of the undesired transmissions the optimum number of streams is one. On the other hand when the interference was limited, increasing the number of streams was shown to be effective [9]. In the following we clarify to ambiguity of the two results. We first prove that single stream transmission is also the optimum scheme for the ERD metric. As the active node density is bounded by the actual node density, we next consider the limited density case. We show that for low enough densities, spatial multiplexing can bring significant performance gain. We also derive a closed form expression for the critical density, below which single stream transmission is not optimal.

The rest of this paper is organized as follows: Section II describes the system model. Section III introduces the novel lower bound and the performance analysis. Section IV presents numerical results and our concluding remarks are given in Section V.

II. SYSTEM MODEL

We assume a decentralized wireless ad-hoc network utilizing a slotted ALOHA protocol (e.g., [4]). Some of the nodes have data that need to be transmitted to specific destinations.
Assuming the operation of a routing mechanism, each message is relayed to its destinations through multiple hops. For simplicity we assume that the receiver of the next hop for each message is located at a fixed distance, \( d \), from the transmitter. Nodes that have data to transmit can access the channel at any time slot with probability \( p \). For any given time slot, the active transmitter distribution is modeled by a two dimensional PPP with a density of \( \lambda = p\lambda_p \) where \( \lambda_p \) is the physical density of nodes.

We assume that each node is equipped with \( N_t \) transmit antennas and \( N_r \) receive antennas. The channel matrix from transmitter \( j \) to receiver \( i \) is denoted by \( H_{ij} \in R^{N_r \times N_t} \) with i.i.d. complex Gaussian distributed entries. Each transmitter is assumed to perform spatial multiplexing of \( K \) streams [10], limited in the range \( 1 \leq K \leq \min (N_r, N_t) \). Note that in the following we are interested in performing scaling analysis of the network performance and hence assume that the users are equipped with large number of antennas.

The received signal at receiver \( i \) is given by:

\[
r_i = \sum_j \sqrt{\rho_j} X_{i,j}^{-\frac{\alpha}{2}} H_{ij} \mathbf{z}_j + \mathbf{n}_i
\]

where \( \rho_j \) is the transmission power of the \( j \)-th transmitter, \( X_{i,j} \) is the distance between the \( j \)-th transmitter and the \( i \)-th receiver respectively, \( \mathbf{z}_j \) is the transmitted signal from transmitter \( j \) and \( \mathbf{n}_i \) denotes the thermal noise with a variance of \( \sigma^2 \). The path-loss factor is denoted by \( \alpha \) and in order to simplify the analysis we further limit our discussion to a path-loss factor within the range \( 2 < \alpha < 4 \).

The precoded signal of the \( i \)-th transmitter is given by:

\[
\mathbf{z}_i = \sum_{k=1}^{K} \mathbf{w}_{i,k} z_{i,k}
\]

where \( z_{i,k} \) and \( \mathbf{w}_{i,k} \) denote the data symbol and the precoding vector of the \( k \)-th data stream for the \( i \)-th transmitter pair, respectively.

The optimal linear receiver, which maximizes the Signal-to-Interference-and-Noise-Ratio (SINR), is the Minimum Mean Square Error (MMSE) receiver [11]. However, the performance analysis of MMSE receivers is very complex and instead we choose to analyze a Partial Zero-Forcing (PZF) receiver. The PZF receiver cancels the interference from its \( M \) closest interferers and uses the rest of signal dimensions for beamforming. For single stream transmission the TC of the PZF receiver was shown to scale as the optimal MMSE receiver [12].

In the slotted ALOHA protocol, transmissions of all active users in each slot are time aligned. In order to perform zero forcing of the interferers the receivers adapt their receiving vectors in the beginning of each slot. Since the ALOHA protocol does not employ any signaling messages or inherent feedback between the receivers and transmitters, the transmitters cannot have knowledge regarding the receive vectors of their paired receivers and hence cannot simultaneously adapt their precoding vectors in an optimal way (i.e., to achieve eigenvalue transmission). Therefore we assume that the transmission vectors are selected in a way that is independent of the channel realizations. For simplicity we set the \( k \)-th entry of the precoding vector, \( w_{i,k} \), to 1 while the rest of its entries are set to zero. The power constraint is enforced by setting \( \rho_j = \frac{1}{K} \forall j \).

The \( N_r \) dimensions of the received vector are allocated to the different schemes. Assuming that the desired streams are also detected using a zero forcing technique the detection of the desired signal requires \( K \) dimension while the interference cancellation requires \( M \cdot K \) dimensions. The rest of the dimensions, denoted by \( L \), are utilized to maximize the power of the desired signal by a receive beamforming scheme. The relation between the allocated dimensions is described by:

\[
L = 1 + N_r - K \cdot (M + 1).
\]

This equality establishes the design tradeoff for choosing the best values of \( M \), \( K \) and \( L \).

Using the shift invariance property, [13], we analyze the performance of the network using a probe receiver. Without loss of generality we assume that the probe receiver is located at the origin. For notational simplicity we drop the probe receiver index and order the interferers’ indices by their distance from the probe receiver in an increasing order, i.e. \( X_1 \leq X_2 \leq X_3 \leq \ldots \). Without loss of generality we assume that \( d = 1 \). Denoting by \( Y_L \) the desired signal power, the Ergodic Rate Density (ERD) of the network with an active user density of \( \lambda \) is given by [7]:

\[
R(\lambda) = \lambda K \cdot E \left[ \log_2 \left( 1 + \frac{Y_L}{\sigma^2 + \sum_{j=M+1}^{\infty} X_j^{-\alpha} V_j} \right) \right] .
\]

### III. PERFORMANCE ANALYSIS

**Theorem 1 (Lower bound):** A lower bound on the ERD of a network with an active user density of \( \lambda \), when each transmitter transmits \( K \) independent spatial stream, \( 1 \leq K \leq \min (N_t, \lfloor N_r / 2 \rfloor) \), each receiver cancels its \( M \) closest transmitters, \( 1 \leq M \leq \lfloor N_r / K \rfloor - 1 \), and the desired channel power, \( Y_L \), distributes as a Chi-square with \( L \) degrees of freedom, is:

\[
R_{LB}^{M,K}(\lambda) \geq R_{LB}^{M,K}(\lambda)
\]

where

\[
R_{LB}^{M,K}(\lambda) = \lambda K \cdot E \left[ \log_2 \left( 1 + \frac{\frac{1}{K} Y_L}{\sigma^2 + C_{\alpha,M,K} \cdot \lambda^Z} \right) \right]
\]

for

\[
C_{\alpha,M} \triangleq \frac{2\pi \frac{1}{2} (M - 2)^{-\frac{1}{2}}}{(\alpha - 2)}.
\]

**Proof of Theorem 1:** We start the proof by presenting the lower bound from [7] on the ERD of a network with a single stream transmission, a path-loss factor in the range \( 2 < \alpha < 4 \).
and an active user density of \( \lambda \), when each receiver cancels its \( M \) closest transmitters and \( M \geq 1 \):

\[
R^M (\lambda) \geq R^M_{\text{LB}} (\lambda)
\]

where

\[
R^M_{\text{LB}} (\lambda) = \lambda \cdot E \left[ \log_2 \left( 1 + \frac{\rho \cdot Y}{\sigma^2 + C_{\alpha,M} \cdot \lambda^\frac{2}{\alpha}} \right) \right]
\]

and

\[
C_{\alpha,M} \triangleq \frac{2\pi^{\frac{2}{\alpha} - 2}}{\alpha - 2} E [\nu V] \left( M - \frac{\alpha}{4} \right)^{\frac{1}{\alpha}}.
\]

In order to lower bound the ERD when utilizing spatial-multiplexing, the lower bound for the single stream case is applied for each stream separately. The \( k \)-th data stream is decoded by a multiplication of the received signal, first by a zero-forcing matrix, \( P_k \), and then by a beamforming vector, \( b_k \). The zero-forcing matrix, \( P_k \), zeroes the signals from the \( MK \) streams of the \( M \) closest interferers in addition to the \( K-1 \) streams of the desired transmitter. The rest of \( L \) dimensions are then used by vector \( b_k \) to maximize the desired signal power by a receive beamforming scheme. Since the channel between each transmitter and each receiver distributes as a complex Gaussian, the desired channel power distributes as a Chi-square with \( L \) degrees of freedom, i.e., \( 2Y_L \sim \chi^2_L \). The channel realizations between an interfering transmitter and the desired receiver (excluding the cancelled interferers) are independent in the matrix \( P_k \) and the vector \( b_k \). Hence, the expectation of the interference power, that results from the transmission of \( K \) streams is \( E[V] = K \). Substituting \( E[V] = K, Y = Y_L \) and \( \rho = 1/K \) conclude the proof.

The following subsections analyse the optimal working point of the network in the interference limited regime, i.e., when \( \sigma^2 \to 0 \).

A. The Optimum Active Users Density Case

In the following subsection we assume that the physical density of the users in the network is large. In this case, by adjusting the transmission probability parameter, \( p \), the density of the active users, \( \lambda \), can be optimized to any desired value.

The following Lemma uses the parameter \( B(\alpha) \triangleq \frac{\alpha}{2} + W \left( \frac{\alpha}{2} e^{-\frac{\alpha}{2}} \right) \) where \( W(\cdot) \) denotes the principal branch of the Lambert \( W \) function defined as the inverse function of \( f(w) = we^w \).

**Lemma 1:** For large values of \( N_r \), the parameters which optimizes the ERD, \( \lambda^*, K^*, M^* = \max_{\lambda,K,M} R^{M,K}_{\text{LB}} (\lambda) \), are given by:

\[
K^* = 1,
\]

\[
M^* = \left( 1 - \frac{2}{\alpha} \right) N_r + \frac{1}{2},
\]

\[
\lambda^* = \left( \frac{\alpha - 2}{\alpha - 4} \right) \cdot \left( N_r + \frac{\alpha}{4} \right)^{\frac{1}{\alpha} - \frac{2}{\alpha}} \cdot \left( N_r - \frac{\alpha}{4} \right)^{\frac{2}{\alpha}} \left( \exp (B(\alpha))^{-1} \right)^{\frac{2}{\alpha}}.
\]

and the optimal ERD is proportional to \( N_r \), with a gain factor of:

\[
G(\alpha) \triangleq \lim_{N_r \to \infty} \frac{R^M_{\text{LB}} (\lambda^*, K^*, M^* \mid \lambda)}{N_r} = \frac{\left( \frac{\alpha - 2}{\alpha - 4} \right) \cdot B(\alpha)}{\ln(2) \cdot \exp (B(\alpha))^{-1} \frac{2}{\alpha}}.
\]

Similar linear scaling as the number of antennas was obtained in [12] and [8] for the TC metric. Note that in [12] and [8] the SIR was not optimized, preventing the derivation of an exact gain factor.

Note that the value in (12) is generally not an integer, and for practical considerations additional rounding process is required. The same hold also for the optimal number of dimensions allocated for beamforming,

\[
L^* = 2 \alpha N_r - \frac{1}{2}.
\]

Also note that for \( \alpha \) in the range 2 to 4 the ERD gain factor, (14), is an increasing function in the range 0 to 0.18.

**Proof of Lemma 1:** Substituting \( x = \lambda \cdot L \cdot \frac{2}{\alpha} K^* C_{\alpha,M}^2 \) into (6) results in:

\[
\max_x R^{M,K}_{\text{LB}} (\lambda) = K^{1-\frac{2}{\alpha}} \cdot L^{\frac{2}{\alpha}} C_{\alpha,M}^{-2} \cdot \max_x E \left[ \log_2 \left( 1 + \frac{\frac{1}{4} Y_L}{x^2} \right) \right].
\]

For large values of \( L \) the expression \( 1/L \cdot Y_L \) converges to 1 and hence the second line of (16) has negligible impact on the optimal parameters. By substituting (3) into (7) we see that the right hand side of the first line of (16) can be written as:

\[
\left( \frac{2\pi^{\frac{2}{\alpha} - 2}}{\alpha - 2} \right)^{\frac{2}{\alpha}} \left( 1 + N_r - L \cdot \left( 1 + \frac{\alpha}{4} \right) \right)^{1-\frac{2}{\alpha}} \cdot L^{\frac{2}{\alpha}}.
\]

AS can be observed, for all values of \( L \), (17) is a monotonically decreasing function in \( K \) and hence the optimum number of streams is \( K^* = 1 \), i.e., the single stream case.

We next optimize (17) for \( K = 1 \) as function of \( L \). Substituting \( K = 1 \) and comparing its derivative to zero results in (15). Substituting (15) into (3) results in (12), and substituting \( \lambda^*, K^*, M^* \) and (15) into (16) results with (14) and concludes the proof.

B. Limited Density

This subsection analyzes the performance for a given density, \( \lambda = \lambda_0 \).

1) Optimum System Parameters: In this case, we wish to find the system parameters, \( (L, M, K) \) that maximize the ERD. As we consider the interference limited regime, the ERD can be written as:

\[
R^*_{\text{LB}} (\lambda_0) = \lambda_0 \max_{K,M} K \cdot E \left[ \log_2 \left( 1 + \frac{\frac{1}{4} Y_L}{L \cdot C_{\alpha,M} \cdot \lambda_0^\frac{2}{\alpha}} \right) \right].
\]
We first focus on finding the optimal value of \( L \) for a given value of \( K \). As the term \( \frac{1}{L} \cdot Y_L \) converges to 1, the optimization of \( L \) effects only the denominator of the term inside the log function. Substituting (3) into (7) leads to:

\[
\frac{K}{L} C_{\alpha,M} = \frac{2\pi^\frac{3}{2} K \left( M - \frac{\alpha}{2} \right)^{1-\frac{\alpha}{2}}}{(\alpha - 2) L} \\
= \frac{2\pi^\frac{3}{2} K \left( 1 + N_r - L \left( 1 + \frac{\alpha}{2} \right) K \right)^{1-\frac{\alpha}{2}}}{(\alpha - 2) L}.
\]

(19)

Evaluating the derivation of (19) with respect to \( L \) and comparing it to zero, \( \frac{\partial}{\partial L} \left( \frac{K}{L} C_{\alpha,M} \right) = 0 \), results in the optimum number of dimensions, allocated to beamforming:

\[
L_K = \frac{2}{\alpha} \left( 1 + N_r - \left( 1 + \frac{\alpha}{2} \right) K \right).
\]

(20)

Substituting (3) into (20) results in the optimum number of cancelled interferers as function of the number of streams:

\[
M_K = \frac{1}{L} \frac{1}{K} \left( 1 - \frac{2}{\alpha} \right) \left( 1 + N_r - K \right).
\]

(21)

For the final stage (21) is substituted into (18) and the optimum number of streams can be solved numerically using the single dimensional optimization:

\[
K^* = \arg \max_K \lambda_0 K \cdot \left[ \log_2 \left( 1 + \frac{Y_L}{C_{\alpha,K} \cdot \lambda_0^\frac{3}{2}} \right) \right]
\]

(22)

where

\[
C_{\alpha,K} \triangleq \frac{2}{\alpha (\alpha - 2)} \left( \pi K \right)^{\frac{3}{2}} \left( \left( \frac{8 - 2\alpha - \alpha^2}{4\alpha} \right) K \right.
\]

\[
\left. + \left( 1 - \frac{2}{\alpha} \right) \left( 1 + N_r \right) \right)^{1-\frac{\alpha}{2}}.
\]

(23)

Note that, as expected, the accumulated interference, \( K \cdot C_{\alpha,M} \cdot \lambda_0^\frac{3}{2} \), is monotonically increasing with \( K \).

2) The Critical Density : We define the critical density, denoted by \( \lambda_{C\alpha} \), as the maximum density in which the spatial multiplexing scheme has a potential performance gain over single stream transmission. Clearly \( \lambda_{C\alpha} < \lambda^* \). Mathematically speaking the critical density of the lower bound is defined by the equality:

\[
\max_M R_{LB}^{M,K = 1} (\lambda_{C\alpha}) = \max_M R_{LB}^{M,K = 2} (\lambda_{C\alpha}).
\]

(24)

For large \( N_r \), the value of \( L_K \) in (20) converges to

\[
L_K \simeq \frac{2}{\alpha} N_r
\]

(25)

and (23) can be simplified into

\[
C_{\alpha,K} \simeq \frac{2}{\alpha (\alpha - 2)} \left( \pi K \right)^{\frac{3}{2}} \left( 1 - \frac{2}{\alpha} \right) N_r^{1-\frac{\alpha}{2}}.
\]

(26)

Since the optimal number of dimensions allocated for beamforming, \( L_K \), growth linearly with \( N_r \) we again assume that the Random Variable (R.V.) \( Y_L/K/L_K \) converges to 1 in the working point of (24). This allows us to drop the expectations. Substituting (18) into (24) results in

\[
\left( 1 + \frac{1}{L} \frac{1}{K} C_{\alpha,2} \cdot \lambda_{C\alpha}^\frac{3}{2} \right) = 1 + \frac{1}{L} \frac{1}{K} C_{\alpha,1} \cdot \lambda_{C\alpha}^\frac{3}{2}
\]

(27)

which is solved by

\[
\lambda_{C\alpha} = \left( \frac{C_{\alpha,2} \left( C_{\alpha,2} - 2C_{\alpha,1} \right)}{C_{\alpha,1}} \right)^{-\frac{2}{\alpha}}.
\]

(28)

Substituting (26) for \( K = 1 \) and 2 into (28) leads to:

\[
\lambda_{C\alpha} = N_r \cdot \left( \frac{\alpha - 2}{2\alpha} \right) \left( 2\alpha^2 - 2 \right)^{-\frac{2}{\alpha}}.
\]

(29)

For \( 2 < \alpha < 4 \) the ratio between (29) and (13) is in the range 0.7 to 0.72, which leads us to state that \( \lambda_{C\alpha} \approx 0.7 \cdot \lambda^* \).

IV. NUMERICAL RESULTS

In the following section we verify the accuracy of the presented results and compare it to random network simulations. The simulations were performed over 5,000 network realizations each with at least 30,000 active nodes. The figures also present the performance of a network that utilizes the optimal parameters of Section III, i.e., \( R_{LB}^{M,K_\star} (\lambda) \). This network referred herein as bound-optimized.

Fig. 1 depicts the maximum ERD, the bound-optimized ERD, the lower bound, (9), and the maximum single stream ERD, \( \max R_{LB}^{M,K=1} (\lambda) \), as a function of the active nodes density, \( \lambda \), for \( \alpha = 3.5 \) and \( N_r = 8 \). As expected the maximum ERD is obtained for \( K = 1 \). Although the analysis was performed for large number of antennas, even for 8 antennas the lower bound is quite accurate. Moreover the performance of the bound-optimized ERD are very close to the optimum ERD, which implies that the optimization parameters can be very useful to perform system optimization. As can be
observed spatial multiplexing gives significant gains when the active node density is low (for example using single stream transmission for $\lambda = 0.09$ results in a 40% loss in the ERD).

Fig. 2 depicts the parameters that optimizes the lower bound for $\alpha = 3.5$ and $N_r = 8$ (see equations (20)-(22)). As expected a growth in the active users density is leading to a decrease in the number of streams and an increase in the number of dimensions that are allocated to beamforming and interference cancellation.

Fig. 3 depicts the calculated gain factor, (14), and the measured gain factor, $\frac{1}{N_r} \cdot \max_{M,K} R^{M,K}(\lambda)$, as a function of the path-loss factor for $N_r = 8$. As can be seen for a path-loss factor of $\alpha = [2.5, 3, 3.5]$ the calculated gain, (14), is equal to $[0.065, 0.106, 0.146]$ while the measured gain is equal to $[0.062, 0.104, 0.142]$ accordingly. Hence, the analytical expression of the gain factor, (14), is very accurate and can be used to evaluate the optimum network performance of MIMO WANETs.

V. CONCLUSIONS

We analyzed the Ergodic Rate Density (ERD) of ALOHA Wireless Ad-Hoc Networks (WANETs) utilizing multiple antennas and presented a novel lower bound on the ERD which converges to the actual ERD for large number of antennas. This bound allows us to analyse the performance of WANETs that utilize spatial-multiplexing, interference-cancellation and beamforming and to present expression for the optimal network parameters.

At the optimum working point, the number of transmission streams was shown to be one. The scaling of the optimum ERD was shown to be proportional to the number of antennas, and the proportionality constant was shown to depend only on the the path-loss factor. We also presented the maximum active users density for which spatial-multiplexing increases the network performance and showed that its equal to 70% from the optimal density.

REFERENCES